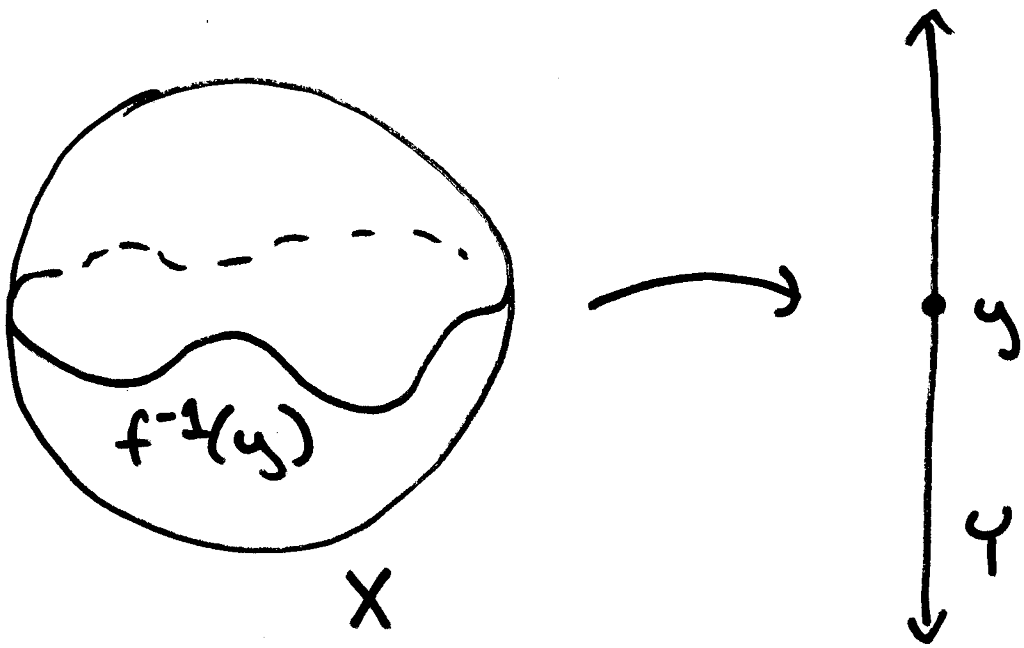
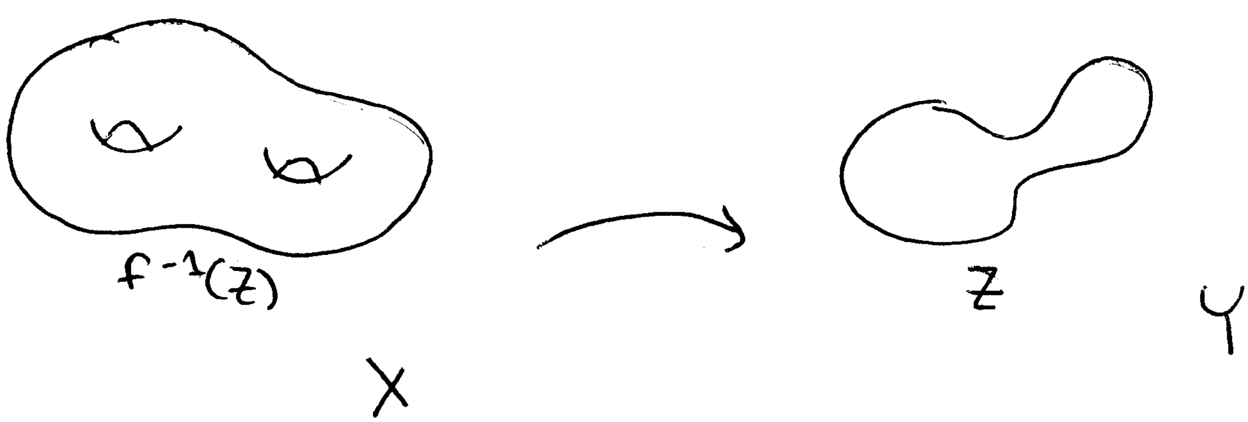
Transversality

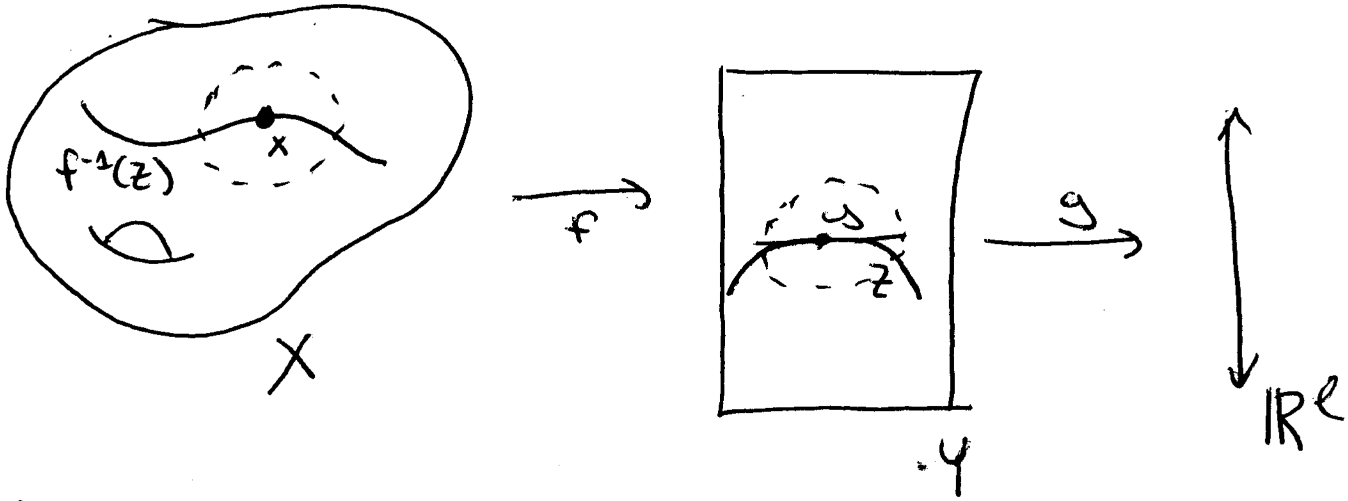
We have considered the case



where we are analyzing the inverse image of a point. What about the inverse image of a submanifold?



Consider the general setup



We know that (locally) around y, Z is the inverse image of $\vec{0}$ under some map $G: Y \to \mathbb{R}^{\operatorname{cod} Z}$. So

$$f^{-1}(Z) = (g \circ f)^{-1}(\vec{0}),$$

which means

$$f^{-1}(Z) \text{ is a submanifold} \iff \vec{0} \text{ is a regular value of } g \circ f.$$

~~When~~

Q: When is $\vec{0}$ a regular value of $(g \circ f)$?

A: When $d(g \circ f)_x : T_x X \to T_{\vec{0}} \mathbb{R}^\ell$ is surjective ~~for all $x \in f^{-1}(\vec{0})$~~ for all $x \in f^{-1}(\vec{0})$.

Linear Algebra Fact 1.

Given a linear map $A : X \to Y$, and a subspace $V \subset X$,

$$\mathrm{Im}_A(V) = \mathrm{Im}_A(\mathrm{Span}(V, \ker A)).$$
$$= \mathrm{Im}_A(V + \ker A).$$

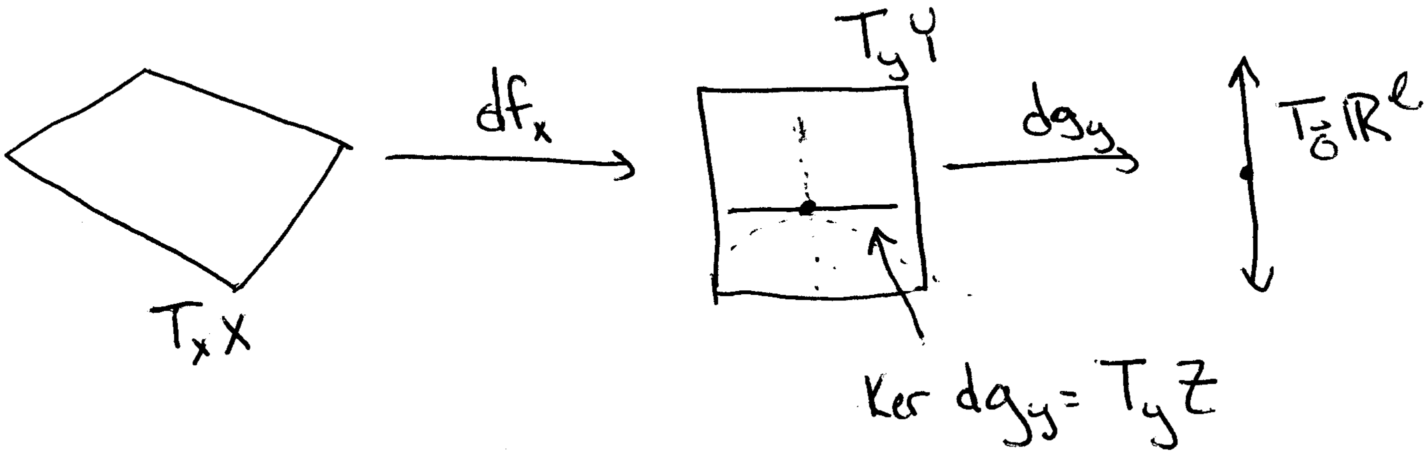
Linear Algebra Fact 2.

Now

$$d(g \circ f)_x = dg_y \circ df_x$$

~~and we~~

So consider



~~In fact, by the local submersion theorem,~~

$$\cancel{(T_yZ)^\perp \cong T_0\mathbb{R}^l}$$

~~a dimension dim Y − dim Z vector space~~

~~isomorphism by dg_y~~

~~also a dimension cod Z vector space~~

So consider

$$\operatorname{Im}_{dg}(\operatorname{Im} df \oplus \ker dg).$$

We know that the dimension of this image in $T_0\mathbb{R}^\ell$ is equal to the rank of dg on $\operatorname{Im} df \oplus \ker dg$.

But this rank is given by

$$\operatorname{rank} dg + \dim \underbrace{\ker dg}_{T_y Z} = \dim(\operatorname{Im} df + \ker dg)$$

We know that

$$dg \circ df \ \text{is surjective onto } \mathbb{R}^\ell \iff \operatorname{rank} dg = \operatorname{cod} Z = \ell$$

$$\iff \dim(\operatorname{Im} df + \ker dg) = \operatorname{cod} Z + \dim Z$$

$$\iff \dim(\operatorname{Im} df + \ker dg) = \dim Y$$

$$\iff \boxed{\operatorname{Im} df + \ker dg = T_y Y.}$$

Definition. We say that a map $f: X \to Y$ is <u>transversal</u> to $Z \subset Y$ if for each $x \in f^{-1}(Z)$,

$$\operatorname{Im} df_x + T_{f(x)} Z = T_{f(x)} Y.$$

The argument above + the preimage theorem tell us

Theorem. If $f: X \to Y$ is transversal to Z then $f^{-1}(Z)$ is a smooth submanifold of X. Further, $\operatorname{cod} f^{-1}(Z) = \operatorname{cod} Z$.

Example.