Morse Theory (completed)

We finished last class with a description of the Morse index of a critical point and a false claim about adding indices.

Here's the truth.

Theorem: Let $C(n)$ be the number of critical points of any Morse function $f : X \to \mathbb{R}$ of Morse index $n$. Then

$$\sum_{n=0}^\infty (-1)^n C(n) = X(x)$$

where $X$ is the Euler characteristic.
In fact, we can prove this using an appealing set of ideas.

Let $X^a = f^{-1}(a)$ and $X^b = f^{-1}(b)$. If there are no critical values of $f$ between $a$ and $b$, then

\[ X^a \cong X^b. \]

diffeo

If there is a single critical point of index $n$ between $a$ and $b$, then

$X_b$ is homotopic to $X_a + \text{an } n\text{-dimensional } \text{cell}$

\[ \rightarrow \]

\[ \Rightarrow \]
Roughly, the Euler characteristic is an alternating sum of the number of \( i \)-dimensional "cells" in a manifold.

For surfaces,

\[
\chi(S) = \text{# vertices} - \text{# faces} + \text{# edges}.
\]
Morse theory was one clever application of Sard’s theorem. Another is the Whitney embedding theorem (easy version).

We start with an n-manifold $X \subset \mathbb{R}^N$ for “some large $N$”. How large does $N$ need to be? Well, $N \geq n$, for sure.

![Klein bottle]

It can be shown (take the 8000 topology course next year!) that the Klein bottle does not embed in $\mathbb{R}^3$.

Whitney showed

Theorem. Any n-manifold $X^n$ has an embedding $f: X \to \mathbb{R}^{2n}$. 
This theorem is hard! But we'll prove the easier theorem.

Theorem. Any $n$-manifold $X^n$ has an embedding into $\mathbb{R}^{2n+4}$.

We start with a construction.

Given $X \subset \mathbb{R}^N$, the set

$$\{ (\vec{x}, \vec{v}) \in \mathbb{R}^N \times \mathbb{R}^N \mid \vec{x} \in X, \vec{v} \in T_{\vec{x}}X \}$$

is called the **tangent bundle** $TX$ of $X$.

**Fact:** If $X \cong Y$, then $TX \cong TY$ by the associated map $(f, df)$.

**Proposition.** If $X$ is a smooth manifold, $TX$ is a smooth manifold with

$$\dim TX = 2 \dim X.$$
Proposition. Every $n$-manifold $X^n$ has an injective immersion $\phi$ into $\mathbb{R}^{2n+1}$.

Proof. We know that for some $N$, there is an injective immersion

$$f : X \to \mathbb{R}^N$$

If $N > 2k+1$, we will find some $\hat{a} \in \mathbb{R}^N$ so that if $\Pi$ is the projection orthogonal to $\hat{a}$, then $\Pi \circ f : X \to \mathbb{R}^{2n+1}$ is still an injective immersion.

Idea: The only thing that could kill injectivity is if some $f(x)$ and $f(y)$ on $\Pi f(X)$ are connected by a line in direction of $\hat{a}$.

Let $h : X \times X \times \mathbb{R} \to \mathbb{R}^N$ be given by

$$h(x,y,t) = t [f(x) - f(y)]$$
Then $\text{Im } h = \text{points on lines which intersect } f(X) \text{ more than once. Since}$

$$\dim(X \times X \times I) = 2n + 1 < \dim \mathbb{R}^N = N,$$

the only regular values of $\phi h$ are points not in $\text{Im } h$. So $\text{Im } h$ is a measure $0$ set of $\mathbb{R}^N$, by Sard's theorem.

Idea: The only thing that could kill immersivity is if $a \in T_x X$ for some $x$.

Again, since

$$\dim T_X = 2n < \dim \mathbb{R}^N = N,$$

we see $\text{Im } g$ is a measure $0$ set of $\mathbb{R}^N$, again by Sard.