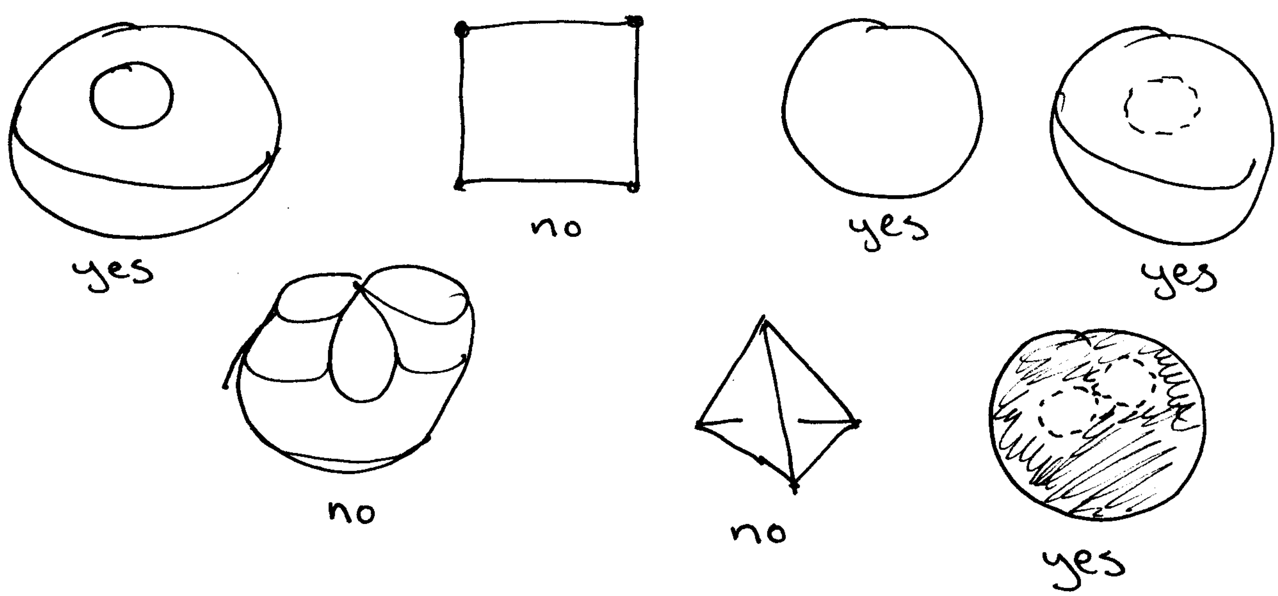
Manifolds with boundary.

We refer to

$$\{(x_1, \dots, x_n) \mid x_n \geq 0\} \quad \text{as} \quad H^n$$

Definition. A subset X of $\mathbb{R}^N$ is called a k-dimensional <u>manifold with boundary</u> if each $x \in X$ has an open neighborhood diffeomorphic to an open set in H^k.

Examples.



Proposition. If X is a manifold with boundary and Y is a manifold without boundary, then

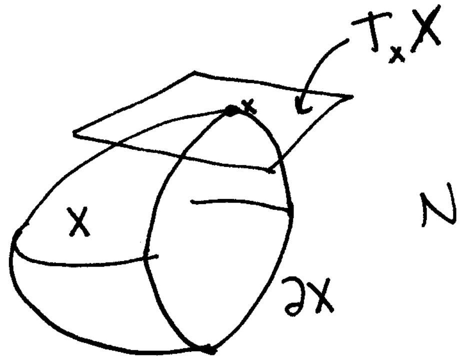
$X \times Y$ is a manifold with boundary

$$\partial(X \times Y) = \partial X \times Y$$

$$\dim(X \times Y) = \dim X + \dim Y.$$

Proof. Easy.

Tangent spaces and derivatives work in the same way for these manifolds.



Note: $T_x X$ is still a k-dimensional linear subspace, even if $x \in \partial X$.

Proposition. If X is a k-dimensional manifold with boundary, then ∂X is a $k-1$ dimensional manifold without boundary.

Proof. ← skipped to save time →

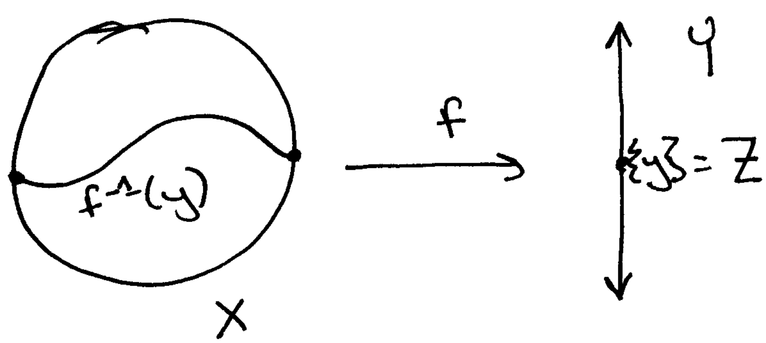
For $x \in \partial X$, $T_x(\partial X) \subset T_x X$. We call

∂f the restriction of f to ∂X

for any $f: X \to Y$. Note

$d(\partial f): T_x(\partial X) \to T_y Y$ is the restriction of df_x

Now suppose



We want to guarantee that $f^{-1}(Z)$ is a submanifold-with-boundary of X.

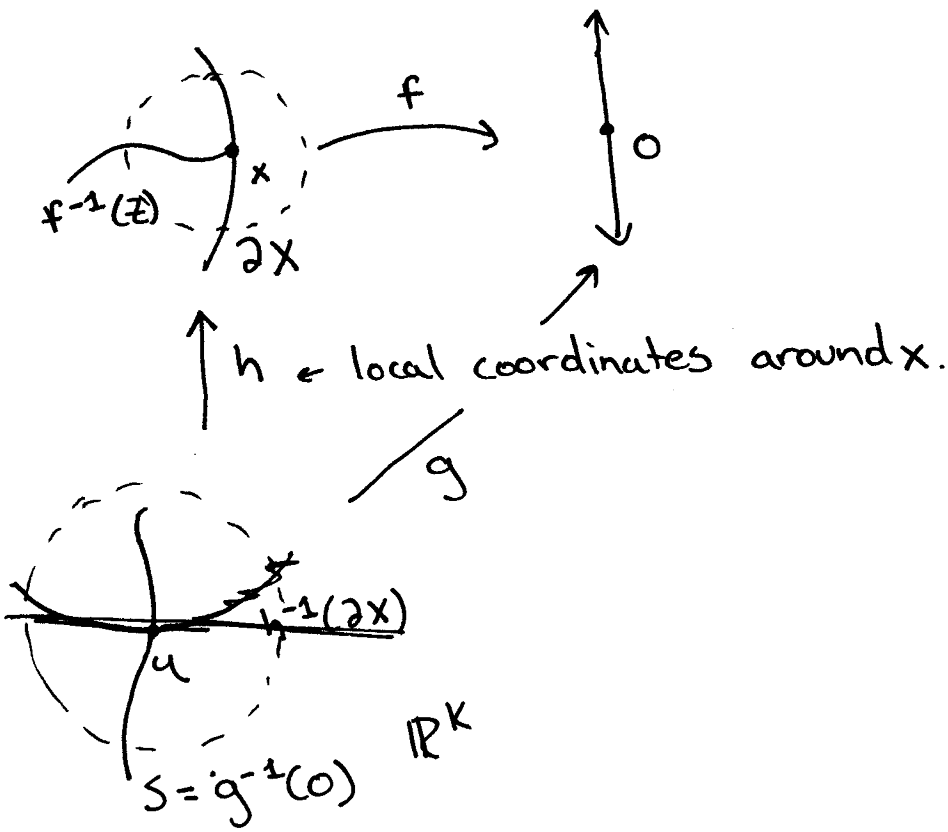
Theorem. If $f: X \to Y$ and $\partial f: \partial X \to Y$ are both transversal to Z, ~~then~~ (and Z has no boundary) then

$f^{-1}(Z)$ is a manifold with boundary $f^{-1}(Z) \cap \partial X$

and $\operatorname{cod} f^{-1}(Z) = \operatorname{cod} Z$.

Proof. On Int(X), everything works as before.

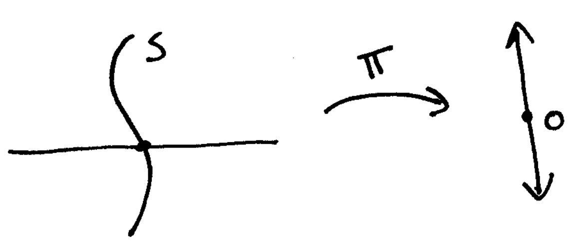
Consider (wlog) the case



By transversality (of f), if we ~~we can~~ extend ~~h~~ h and g to an open neighborhood ~~to~~ of u, $g^{-1}(0)$ is a submanifold S of $\mathbb{R}^K$.

Is $S \cap H^K$ a submanifold with boundary of H^K?

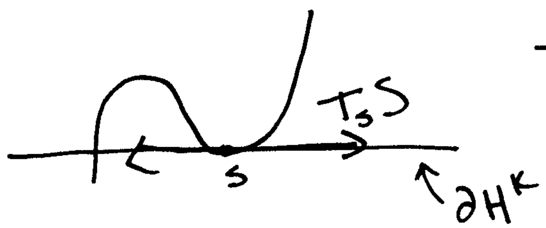
Consider



$\pi(s) = s_K$.

We claim 0 is a regular value for π on S.

Suppose not. Then $\exists\ s \in S$ with $\pi(s) = 0$ and $d\pi_s = 0$.
Since π is linear, $d\pi = \pi$ and this means



$T_s S \subset T_s(\partial H^K)$.

This means that $T_s S = \ker dg \subset T_s(\partial H^K)$.
But $d(\partial g)$ is the restriction of dg to $T_s(\partial H^K)$, so

$d(\partial g)$ and dg have the <u>same</u> kernel at s.

But by transversality $d(\partial g)$ and dg are both surjective onto $\mathbb{R}$. So if each has rank 1,

$$\dim \ker dg = K - \operatorname{cod} Z - 1$$
$$\dim \ker d(\partial g) = (K-1) - \operatorname{cod} Z - 1$$

but these aren't equal! $\times\!\!\times$

So 0 <u>is</u> a regular value for π on S.

Claim. If S is a manifold w/o boundary and $\pi : S \to \mathbb{R}$ is a smooth function with regular value 0, then

$$\{ s \in S \mid \pi(s) \geq 0 \}$$

is a manifold <u>with</u> boundary ~~and~~ $\pi^{-1}(0)$.

Proof. Local submersion theorem at boundary points.

Sard's Theorem. For any smooth map $f : X \to Y$ of a mfld with boundary into a boundaryless manifold Y, almost every $y \in Y$ is a regular value of f and ∂f.

Proof. Easy.

Theory of manifolds w/ boundary.

We first observe:

Theorem. Every compact, connected 1-manifold with boundary is either I or S^1.

Corollary. The boundary of any compact 1-manifold with boundary is an even number of points.

Theorem. If X is a compact manifold with boundary, $\exists$ no smooth map $g: X \to \partial X$ with $\partial g: \partial X \to \partial X$ the identity. (Such a map is called a "retraction".)

Proof. If so, let $z \in \partial X$ be a regular value. $g^{-1}(\{z\})$ is a submanifold of X with boundary. Now $\{z\}$ has codimension $\dim \partial X = \dim X - 1$, so

$g^{-1}(\{z\})$ has codimension $\dim X - 1$

and has dimension 1.

Then $\partial g^{-1}(z) = g^{-1}(z) \cap \partial X = \{z\}$.

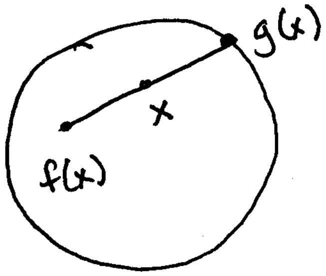
But this is only one point, not an even #.

We can now show

Brouwer Fixed Point Theorem.

Any smooth map $f: B^n \to B^n$ must have a fixed point.

Proof. Suppose not. We build a map g from f



so that g is a retraction, by taking

$g(x)$ = the intersection of the line through $f(x)$ and x with ∂B^n.

if $x \in \text{Int } B^n$ and $g(x) = x$ if $x \in \partial B^n$.

Is g smooth? Well, yes. But how to prove it? Well, the line through $f(x)$ and x looks like

$$L(t) = (1-t) f(x) + t x, \quad t \geq 0.$$

We see $|L| = 1$ (or $L \in \partial B^n$) when

$$(1-t)^2 f(x) \cdot f(x) + 2t(1-t) f(x) \cdot x + t^2 x \cdot x = 1.$$

Gather like terms in t to see that this is a nice quadratic.