Transversality Theorem.

Suppose \( F : X \times S \to Y \) is a smooth map of manifolds where only \( X \) has boundary and let \( Z \subseteq Y \) be a submanifold without boundary.

If \( F, \partial F \) transversal to \( Z \), then for almost any \( s \in S \), \( f_s = F(-,s) \) and \( \partial f_s \) are transversal to \( Z \).

Proof. We know

\[
W = F^{-1}(Z) \text{ is a mfld with boundary}
\]

\[
W \cap \partial(X \times S).
\]

Consider

\[
\pi : X \times S \to S, \quad \pi(x,s) = s.
\]
Claim. If $s$ is a regular value of $\pi|_W$ and of $\pi|_Ω$, then $f_s \pi|_Z$.

Take $(x,s)$ so that $f_s(x) = z \in Z$. We must show that

$$d(f_s)_x T_{(x,s)}(X \times S) + T_z Z = T_z Y.$$ 

or, for any $\hat{a} \in T_z Y$, we must show:

$$\exists \hat{a} \in d(f_s)_x T_{(x,s)}(X \times S) \text{ so that } \hat{a} - \hat{b} \in T_z Z.$$ 

We know that for this $\hat{a}$, $\exists b \in T_{(x,s)} X \times S$ so that

$$\hat{a} - dF_{(x,s)}(\hat{b}) \in T_z Z$$

(transversality of $F$ w.r.t. $Z$). Now

$$T_{(x,s)} X \times S = T_x X \times T_s S$$
\[ \hat{b} = (\hat{w}, \hat{e}) \text{ for } \hat{w} \in T_x X, \hat{e} \in T_{sS}. \]

If \( \hat{e} = 0 \), then \( \hat{w} \) would be our \( \hat{v} \) since

\[ dF_{(x,s)}(\hat{w},0) = df_s(\hat{w}). \]

Plan: Modify \( \hat{b} \) to cancel \( \hat{e} \).

We know that

\[
d\Pi_{(x,s)} : T_{(x,s)}(X \times S) \rightarrow T_{sS} \\
T_x X \times T_{sS}
\]

is just the projection \( (\hat{w}, \hat{e}) \rightarrow \hat{e} \). But \( d\Pi \) maps \( T_{(x,s)} W \) onto \( T_{sS} \) since \( \# s \) is a regular value of \( \Pi|_W \).

Thus \( \exists \) \((\hat{w}, \hat{e})\) so that \((\hat{w}, \hat{e}) \in T_{(x,s)} W\).
Observe that $F(\omega) \in \mathcal{Z}$, so
\[ dF_{(x,\omega)}(\tilde{\nu}, \tilde{e}) \in T_{\tilde{z}}\mathcal{Z}. \]

We claim
\[ \tilde{\nu} = \tilde{\omega} - \# \tilde{u} \]
has the property that
\[ a - df_\gamma(\tilde{\nu}) \in T_{\tilde{z}}\mathcal{Z}. \]

We check
\[
a - df_\gamma(\tilde{\nu}) = a - dF_{(x,\omega)}(\tilde{\nu}, 0)
\]
\[= a - dF_{(x,\omega)}(\tilde{\omega} - \tilde{u}, \tilde{e} - \tilde{e}')
\]
\[= a - \left[ dF_{(x,\omega)}(\tilde{\omega}, \tilde{e}) \right] + dF_{(x,\omega)}(\tilde{u}, \tilde{e})
\]
\[= \underbrace{a - dF_{(x,\omega)}(\tilde{u})}_{\text{in } T_{\tilde{z}}\mathcal{Z} \text{ by construction of } \tilde{u}} + \underbrace{dF_{(x,\omega)}(\tilde{u}, \tilde{e})}_{\text{in } T_{\tilde{z}}\mathcal{Z} \text{ by construction of } (\tilde{u}, \tilde{e})}
\]
\[ \in T_{\tilde{z}}\mathcal{Z}. \]
Since Sard's theorem says almost every \( s \in S \) is a regular value, we're done.

Now imagine

If we build

\[ F : X \times \mathbb{R}^3 \to Y \] by \( F(x, \tilde{z}) = f(x) + \tilde{z} \)

then \( F \) is a submersion onto \( Y \) so \( F \) is always transverse to any \( Z \subset Y \).
By the Theorem above, almost every variation \( z \in \mathbb{R}^3 \) makes the map \( f(x) \) transverse to \( Z \).

Idea: Transverse maps are the only observable maps.

What if \( Y \) is not Euclidean space?
We need to pretend it is...

- Neighborhood theorem.