Winding Numbers and more!

Suppose we have some compact, connected $n-1$ manifold contained in $X$ and a smooth map $f : X \to \mathbb{R}^n$.

![Diagram of a compact, connected manifold being mapped into $\mathbb{R}^n$.]

We want to consider how $f(x)$ wraps in $\mathbb{R}^n$, so pick some $z \notin f(x)$ and consider

$$u(x) = \frac{f(x) - z}{|f(x) - z|} : X \to S^{n-1}$$

(this is the trick we used in the Fundamental Theorem of algebra proof last time.). We know

$\deg u$ is a homotopy invariant of $u$. 

Definition. \( \text{deg}_z u \) is called the \( \text{mod} \ 2 \) winding number of \( f \) around \( z \), and this is written
\[ W_z (f, z). \]

We first claim

\[ \begin{align*}
 & \hspace{1cm} \begin{array}{c}
 \text{Diagram 1} \\
 \text{Diagram 2} \\
 \text{Diagram 3}
 \end{array}
\end{align*} \]

Theorem. Suppose \( X = \mathbb{R}^D \) and let \( F: D \to \mathbb{R}^n \) extend \( f: X \to \mathbb{R}^n \). If \( z \) is a regular value of \( F \) and \( z \notin f(x) \), then
\[ W_z (f, z) = \# \text{ of points in } F^{-1}(z). \]

Proof. We observe that
\[ W_z (f, z) = \text{deg}_z u = I_x (u, 3 \mathbf{v} \mathbf{3}) \]
for a direction \( u \in S^{n-1} \). But if \( u \) extends to \( D \), then \( I_x (u, 3 \mathbf{v} \mathbf{3}) = 0 \) by our Boundary Theorem from last class.
So if \( z \notin F(D) \), we're done. Suppose \( z \in F(D) \). By stack of records, there are open sets in \( D \) called \( V_1, \ldots, V_n \) mapping diffeomorphically to open set \( U \) containing \( z \).

Now if we take little balls \( B_i \) around each of the points in \( F^{-1}(z) \), then \( \omega \) does extend to \( D - (B_1 \cup \ldots \cup B_n) \), so we can define

\[
\omega(x) = \frac{F(x) - z}{|F(x) - z|}
\]

and the collection of maps \( \omega, \omega_1, \ldots, \omega_n \) extend to \( D - (B_1 \cup \ldots \cup B_n) \).
But this means that for \( v \in S^2 \),

\[
I_2(u, \xi v^3) + I_2(u_1, \xi v^3) + \ldots + I_2(u_n, \xi v^3) = 0 \mod 2
\]
or if \( f_i = F |_{\partial B} \)

\[
\omega_z(f_i, z) = \omega_z(f_2, z) + \ldots + \omega_z(f_n, z) \mod 2.
\]

Since \( F : B_i \to U \) is a diffeomorphism, by taking the ball small enough in \( D_3 \), we can ensure \( \omega_z(f_i, z) = 1 \) for all \( i \), proving the Thm.

We can use this to prove:

Jordan-Brouwer Separation Theorem.

Given a compact, connected \( n-1 \) manifold \( X \subset \mathbb{R}^n \), \( \mathbb{R}^n - X \) a consists of two connected open sets, the "inside" \( I \), whose closure is a compact \( n \) manifold with boundary \( X \) and the "outside" \( O \).
The proof is a long and glorious homework assignment, so I’ll give only the basic idea:

\[ I = \text{all } z \text{ with } \omega^2(x, z) \neq 0 \]
\[ O = \text{all } z' \text{ with } \omega^2(x, z') = 0 \]

Next time we’ll prove

**Borsuk-Ulam Theorem.** Let \( f : S^k \to \mathbb{R}^{k+1} \) be a map that avoids \( 0 \in \mathbb{R}^{k+1} \). Suppose \( f \) is odd

\[ f(-x) = -f(x). \]

Then \( \omega^2(f, 0) = 1 \).