Borsuk-Ulam Theorem.

Let \( f : S^k \to \mathbb{R}^{k+1} \) be a smooth map which avoids the origin and has
\[
f(-x) = f(x) \quad \text{for all } x \in S^k.
\]
Then \( \omega_2(f, 0) = 1 \).

Proof: We will prove it for \( k=1 \) in homework. So assume the theorem for \( k-1 \), and take a map \( f : S^k \to \mathbb{R}^{k+1} - \{0\} \).

Let \( g : S^{k-1} \to \mathbb{R}^{k+1} - \{0\} \) be the restriction of \( f \). Consider the maps
\[
\frac{g}{|g|} : S^{k-1} \to S^k \quad \frac{f}{|f|} : S^k \to S^k
\]
and choose \( \tilde{a} \in S^k \) so that \( \tilde{a} \) is a regular

value for both maps (Sard).

This means that \( \hat{a} \notin \text{Im}(\varphi) \) (the dimension of \( S^{k-1} \) is too low) and that if the line \( l \) is \( \text{span}(\hat{a}) \subset \mathbb{R}^{k+1} \), then \( f \cap l \), \( g(S^k) \cap l = \emptyset \).

Now
\[
\omega_2(f,0) = \deg_2 \left( \frac{f}{|f|} \right) = \# \text{ pts in } \left( \frac{f}{|f|} \right)^{-1}(a) \mod 2.
\]

By symmetry,
\[
\frac{1}{n} \# \text{ pts in } \left( \frac{f}{|f|} \right)^{-1}(a) = \frac{1}{2} \# \text{ pts in } f^{-1}(l).
\]

Now on the upper hemisphere, let \( f_+ \) be the restriction of \( f \). Consider

the composition \( \Pi \circ f_+ \) where \( \Pi \) is orthogonal projection in the direction \( \hat{a} \).
We note

(a) On $S^K$, $\pi\circ g(-x) = \pi(-g(x)) = -\pi(g(x))$
   by symmetry of $g$ and linearity of $\pi$.

(b) Since $\mathbb{R} \cap \text{Im}(g) = \emptyset$ (remember $\mathfrak{A}$ was a regular value for $g/|g|$), we have $\pi\circ g$ avoids the origin in $\mathbb{R}^K$.

By inductive hypothesis, $\omega_z(\pi\circ g, 0) = 1$. But $\pi\circ g = \pi(\pi_{\text{of}_+})$, so by our theorem last time,

$\# \text{ pts in } (\pi_{\text{of}_+})^{-1}(0) = \omega_z(\pi\circ g, 0) = 1 \mod 2$

Thus

$1 = \# \text{ pts in } (\pi_{\text{of}_+})^{-1}(0)$

$= \frac{1}{2} \# \text{ pts in } (\pi_{\text{of}})^{-1}(0)$

(by symmetry)

$\frac{1}{2} \# \text{ pts in } f^{-1}(\mathfrak{A})$

(defn of $\pi$)

$= \omega_z(f, 0)$

(by symmetry)

$= \omega_z(f, 0)$. (defn of $\omega_z$)
We can already conclude:

**Theorem.** If \( f : S^k \to \mathbb{R}^{k+1} \) has \( f(-x) = -f(x) \) for all \( x \), then \( \text{Im} \, f \) intersects every line through the origin at least once.

**Proof.** Let \( \ell \) be the line chosen, then \( \omega_2(f,0) = \frac{1}{2} \# \text{pts in } f^{-1}(\ell) = 0 \). \( \Box \)

**Example.**

![Diagram](image)

**Theorem.** Any \( k \) smooth functions \( f_1, \ldots, f_k \) on \( S^k \) with \( f_i(x) = -f_i(x) \) must have a common zero.

**Proof.** Consider \( f : S^k \to \mathbb{R}^{k+1} \) given by

\[
f(x) = (f_1(x), \ldots, f_k(x), 0)
\]

and let \( \ell \) be the \( x_{k+1} \) axis.
Theorem.

Consequence. Any collection of $K$ odd homogenous polynomials in $K+1$ variables has a common root.

We can also get the (remarkable) theorem

Theorem. For any $K$ smooth functions on $S^K$ there are a pair of antipodal $p, -p$ so that $g_1(p) = g_1(-p), \ldots, g_K(p) = g_K(-p)$.

Proof. Let $f_i(x) = g_i(x) - g_i(-x)$. $p$ is the common zero of the $f_k$.

Next class we introduce orientation!