MATH 4600

Midterm Exam II

April 4, 2019

NAME (please print legibly): ____

Your University ID Number: ____

Please complete all questions in the space provided. You may use the backs of the pages for extra space, or ask me for more paper if needed. This exam will be graded on:

- Correctness of computations.
- Clarity of explanation of procedure.
- Correctness of procedure.

A correct answer obtained using an incorrect or poorly explained procedure will not be graded for full credit. Please feel free to write as much as you like. Work carefully, and try to complete the problems you find easier before going back to the harder ones. Good luck!

QUESTION	VALUE	SCORE
1	15	
2	15	
3	20	
4	20	
5	10	
6	25	
7	15	
TOTAL	120	

1. (15 points) Suppose that X and Y are discrete random variables. State the definition of the probability mass function $p_X(x)$. (5 pts)

State the definition of the **joint** probability mass function $p_{X,Y}(x, y)$. (5 pts)

State the definition of the **conditional** probability mass function $p_{X|Y}(x|y)$. (5pts)

2. (15 points)

Suppose that you flip two fair coins. The random variable X is the total number of heads and the random variable Y is the number of heads on the first flip (that is, $Y = 1$ if the first coin is heads and $Y = 0$ if the first flip is tails).

List the possible values x for the r.v. X and compute the probability mass function $p_X(x)$ for each of these possible values. (3pts)

List the possible values y for the r.v. Y and compute the probability mass function $p_Y(y)$ for each of these possible values. (2pts)

Find the joint probability mass $p_{X,Y}(x, y)$ for all the pairs (x, y) of values for the two distributions. (5pts)

Find the conditional probability mass $p_{X|Y}(x|y)$ for all the pairs (x, y) of values for the two distributions. (5pts)

3. (20 points) There are three equivalent definitions of independence for two random variables X and Y in terms of the probability mass function, the CDF, and the conditional mass. Give as many of these definitions as you remember (15 pts, 5 for each one):

Are the random variables X and Y from problem 2 independent or dependent? Prove it.
(1pt: intuit the right answer, 4 pts: justify your intuition using one of the definitions above)

4. (20 points) The weather in the city of Cambridge, England is either rainy (R) or foggy (F). Let X be the random variable "day of the week" (numbered 1-7) and Y be the indicator variable for "rain" (that is, $Y = 1$ if the day is rainy, $Y = 0$ if the day is foggy). The **joint** distribution of days of the week X and weather Y is given by

	Sun ($X = 1$)	Mon (2)	Tue (3)	Wed (4)	Thu (5)	Fri (6)	Sat (7)
R ($Y = 1$)	$\frac{3}{28}$	$\frac{3}{28}$	$\frac{2}{28}$	$\frac{2}{28}$	$\frac{2}{28}$	$\frac{3}{28}$	$\frac{3}{28}$
F ($Y = 0$)	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{2}{28}$	$\frac{2}{28}$	$\frac{2}{28}$	$\frac{1}{28}$	$\frac{1}{28}$

Find the **marginal** distributions $p_X(x)$ and $p_Y(y)$. (5pts)

Compute the probability of the event "rain" in Cambridge. Explain your computation in terms of joint and marginal distributions. (5pts)

Suppose you wake up with your curtains open in Cambridge and observe that it's raining outside. Find the probability that it's Saturday. Explain your computation in terms of conditional probability. (5pts)

Suppose you wake up with your curtains closed in Cambridge and are sure that it's Saturday. Find the probability that it's raining. Explain your computation in terms of conditional probability. (5pts)

5. (10 points) At breakfast on Monday morning, your roommate "Bucky" is scowling. You know that when Bucky is mad at you, he scowls 90% of the time. On the other hand, Bucky is somewhat cranky and scowls 60% of the time when he's not mad at you. You would like to estimate the probability that Bucky is mad at you, given the scowl. Here are two possible scenarios:

Scenario 1. Thinking back, you don't remember anything unusual about the weekend, and decide that your prior estimate of the probability that Bucky is mad at you is 10%. Use Bayes' theorem to compute an updated estimate of the probability that Bucky is mad at you, given that he is scowling. (5pts)

ANSWER: ____________________

Scenario 2. Thinking back, you remember that you spilled cheese dip on Bucky's bionic arm on Saturday, and decide that your prior estimate of the probability that he is still mad at you on Monday morning is 70%. Use Bayes' theorem to compute an updated estimate of the probability that Bucky is mad at you, given that he is scowling. (5pts)

ANSWER: ____________________

6. (25 points) The random variables X_1, X_2, X_3 are produced by rolling 3 different (fair, 6 sided) dice. In a casino game, you can win money in several different ways depending on the values of the X_i:

1. (sixers) 6$ for each '6' that is rolled,
2. (triple-six) an additional 666$ for rolling all 6's,
3. (steps) another 123$ for rolling $X_1 = 1, X_2 = 2, X_3 = 3$.

Let W_1 be the random variable given by your winnings from "sixers", W_2 be your winnings from "triple-six", and W_3 be your winnings on "steps".

Are the r.v.'s X_1, X_2 and X_3 independent? Justify your answer, using the definition of independence or a theorem about independence. (5pts)

Are the r.v.'s W_1, W_2 and W_3 independent? Justify your answer using the definition of independence or a theorem about independence. (5pts)

State the definition of the expected value $\mathcal{E}(X)$ of a random variable X.(5pts)

Find the expected values $\mathcal{E}(W_1)$, $\mathcal{E}(W_2)$ and $\mathcal{E}(W_3)$. (5pts)

Find the expected value of your total winnings. Be sure to justify your answer carefully, either with the definition of expectation or a theorem. (5pts)

7. (15 points) **(Graduate Students)** A certain event A has probability p at each of a (potentially infinite) number of independent trials.

Let X_k be the number of times A is observed in the first k trials. Find $\mathcal{E}(X_k)$. Justify your answer. (5pts)

Let Y_k be the number of trials required to observe A a total of k times. Find $\mathcal{E}(Y_k)$. Prove it. (10pts)

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