Larger Lefschetz Numbers.

We ended last class with a discussion of fields on surfaces and their associated fixed points. We used this to compute $\chi(S)$ for surfaces in $\mathbb{R}^3$.

Theorem. If $S$ is a (compact, smooth, orientable) surface of genus $g$, then $\chi(S) = 2 - 2g$.

What about other fixed points?
These fixed points are not Lefschetz! But by our previous prop, a homotopic map has only Lefschetz fixed points.

Splitting Theorem. Let $U$ be a neighborhood of a fixed point $x$ which contains no other fixed points of $f$. Then a homotopy $f_t$ of $f$ which if outside $U$ so that $f_t$ has only Lefschetz fixed points and each $f_t = f$ outside some compact subset of $U$. 
We now want to define:

Definition. If \( x \) is an isolated fixed point of \( f: \mathbb{R}^k \to \mathbb{R}^k \), we let \( f_1 \) be a local homotopy with only Lefschetz fixed points and

\[
L_x(f) = \sum_{f_1(x) = x} L_x(f_1)
\]

Here is a surprising idea: On \( \mathbb{R}^k \), we can define a map

\[
g_x(z) = \frac{f(z) - z}{|f(z) - z|}
\]

near any isolated fixed point \( z \) of \( f \). Take a sphere \( S^{k-1} \) centered at \( z \) and consider

\[
g_x \in S^k : S^k \to S^k
\]

Definition. \( L_x(f) = \deg g \).
Proof.

Assume \( f: \mathbb{R}^k \to \mathbb{R}^k \) fixes only \( \bar{0} \) in \( U \).
Now choose a bump function \( \Phi \) so that

\[
\Phi \text{ is smooth} \quad \checkmark
\]
\[
is 1 \text{ in a neighborhood of } 0 \text{ contained in a compact subset } K \text{ of } U
\]
\[
is 0 \text{ outside } K
\]

Our idea is to choose some vector \( \bar{v} \) so that

\[
f_t(x) = f(x) + t \Phi(x) \bar{v}
\]

works.

Claim. If \( \bar{v} \) is really small, then \( f_t \) has no fixed points outside \( V_t \) in \( U \).

Consider \( K-V \). This is compact, and \( f \) has no fixed points on it, since \( K-V \subseteq U \), and \( 0 \notin K-V \). So \( |f(x)-x| > c > 0 \) on \( K-V \).
Choose \( |\bar{v}| < c/2 \). Since \( t\Phi \leq 1 \), this works.
Outside \( K \), \( f_t = f \), so there are no fixed points.
By Sard's theorem, we can pick \( \tilde{V} \) close to 0 so that \(-\tilde{V}\) is a regular value for
\[ x \mapsto f(x) - \tilde{V} x. \]

Now suppose \( x \) is a fixed point of \( f_\tilde{V} \).
We know \( x \in V \), so
\[ f_\tilde{V}(x) = f(x) + \tilde{V}. \]

Thus
\[ d(f_\tilde{V})_x = df_x. \]

Now \( x \) is Lefschetz for \( f_\tilde{V} \) \iff \( df_\tilde{V})_x - I \) nonsingular.
But
\[ d(f_\tilde{V})_x - I = df_x - I \]
\[ = d((f(x) - x)_x) = \text{nonsingular, since} \]
\[ f_\tilde{V}(x) = x \implies f(x) - x = -\tilde{V} \]
and \(-\tilde{V}\) is a regular value for \( f(x) - x \).

Now in general, take local coordinates.
These two ideas agree!

Example: do calculations.

Proposition. At Lefschetz fixed points, the numbers agree.

Proof. As usual, let $X = \mathbb{R}^n$ and suppose $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ has 0 as a Lefschetz fixed point.

We can write

$$f(x) = Ax + \epsilon(x), \text{ where } A \approx df_0$$

near the origin. Since $A-I$ is an isomorphism, it maps the unit sphere to an ellipsoid at least $c > 20$ from 0. By linearity,

$$(A-I) z > 12c \text{ from origin.}$$
But then choosing $B$ around $\delta$
small enough so $\frac{|e(z)|}{|z|} < \frac{\delta}{2}$ ($e(z)$ is 2nd order)
we see that on $B$, if $f_k(z) = e(z)z$

$$f_k(z) = Az + te(z),$$

then for all $t$ in $[0,1]$, $|f_k(z) - z| = |Az + te(z) - z|$

$$\geq |Az - z| - t|e(z)|$$

But $|Az| < c|z|^2$, $|e(z)| < c\delta^2/2$, so for small

$$> c|z| - c|z|^2/2 > 0$$

So we have $|f_k(z) - z| > 0$. 
This means that $f_t$ is a homotopy between maps $\phi$ of maps

$$g_t(z) = \frac{f_t(z) - z}{|f_t(z) - z|}$$

defined on $B$ a small $S^{n-1}$ around $\bar{D}$ to $S^{n-1}$. But then

$$\deg g_2(z) = \text{our def. of Lefschetz degree}$$

$$\deg g_0(z) = \deg \frac{(A-I)z}{|A-I|z}.$$

Fact. Any linear isomorphism $B: \mathbb{R}^n \to \mathbb{R}^n$ is homotopic to $I$ if $\det B > 0$, through isomorphisms and to a reflection if $\det B < 0$. 
In the first case, \( \deg \frac{A-I}{|A-I| z} = \deg \frac{z}{|z|} = 1 \),
in the second, \( \deg \frac{(-z_1, z_2, \ldots, z_n)}{|z|} = -1 \).

Cool! We can now define Lefschetz number again in a more insightful way.

Definition. Let \( f: X \to X \) be a map with finitely many fixed points \( x_i \) on a compact manifold

\[
L(f) = \sum_{x_i} L_{x_i}(f),
\]

where

\[
L_{x_i}(f) = \deg \frac{f(x)-x}{|f(x)-x|} : \text{Ball around } x_i \to S^{\dim X-1}.
\]
Try some examples!!

Lefschetz number 2?