Differential forms (IV)

We have now defined a space of alternating p-tensors $\Lambda^p(V^*)$ for any vector space $V$.

We can now define differential forms!

Definition. Let $X$ be a smooth manifold. A $p$-form on $X$ is a function $\omega$ which assigns to each $x \in X$ an alternating $p$-tensor $\omega(x)$ on $T_x X$.

Some properties of forms are immediate:

1. forms can be added and scalar-multiplied pointwise

2. forms can be wedged pointwise, and if $\omega$ is a $p$-form, $\Theta$ is a $q$-form

$$\omega \wedge \Theta = (-1)^{pq} \Theta \wedge \omega$$
Examples.

Any function $\varphi: X \to \mathbb{R}$ is a 0-form.

Given a smooth $\varphi: X \to \mathbb{R}$, the differential $d\varphi_x: T_x X \to \mathbb{R}$ is a linear functional (or alternating 1-tensor) on each $T_x X$. Thus the differential defines a 1-form $d\varphi$.

The coordinate functions

$$x_i: \mathbb{R}^k \to \mathbb{R} \quad \text{define 1-forms} \quad dx_i$$

$$x_k: \mathbb{R}^k \to \mathbb{R} \quad \text{define 1-forms} \quad dx_k$$

We see that at any point $z \in \mathbb{R}^k$,

$$(dx_j(z))(\left(\begin{array}{c} \nu_1 \\ \vdots \\ \nu_k \end{array}\right)) = \nu_j$$

So these 1-forms define a standard basis for p-forms on $\mathbb{R}^k$: if $I=i_1, \ldots, i_p$ then

$$dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

and the basis consists of all $dx_I$ with increasing sequences $I$. 

Proposition. Every p-form on \( U \subset \mathbb{R}^k \) can be written uniquely as

\[
\sum f_I \, dx_I
\]

where the \( f_I \) are functions on \( U \).

We now do a sanity check:

Lemma. Given a smooth \( \Phi: U \to \mathbb{R} \),

\[
d\Phi = \sum \frac{\partial \Phi}{\partial x_i} \, dx_i
\]

Proof. Both sides are linear functionals on \( \mathbb{R}^k \). So we check for each \( \vec{v} \in \mathbb{R}^k \),

\[
d\Phi(\vec{v}) = \langle \nabla \Phi, \vec{v} \rangle
\]

\[
= \sum \frac{\partial \Phi}{\partial x_i} \, v_i
\]

\[
= \sum \frac{\partial \Phi}{\partial x_i} \, dx_i(v_i).
\]
The transpose operation of last class induces a natural way to transfer forms under smooth maps.

Construction. If \( f: X \to Y \) is a smooth map and \( \omega \) is a \( p \)-form on \( Y \), then we define

\[
(f^* \omega)(x) = (df_x)^*(\omega(f(x)))
\]

an alternating \( p \)-tensor on \( T_x X \)

So the pullback form \( f^* \omega \) on \( X \) is given by the tensor field above. In words, given \( v_1, \ldots, v_p \in T_x X \)

\[
f^* \omega(v_1, \ldots, v_p) = \omega(df_x v_1, \ldots, df_x v_p).
\]
We claim

Proposition.

\[ f^*(w_1 + w_2) = f^*(w_1) + f^*(w_2) \]
\[ f^*(w \land \theta) = f^*(w) \land f^*(\theta) \]
\[ (f \circ h)^* \omega = h^* f^* \omega. \]

Proof. The previous explanation for \( f^* \) makes the first two easy. The third is really the statement that given \( X \subseteq Y \subseteq Z \)

\[ (df_Y \circ dh_{f(X)})^* = (dh_{f(X)})^* (df_Y)^* \]

which is easy to check.

We now want to write \( f^* \) explicitly in coordinates.
Given \( U \subset \mathbb{R}^k \) and \( V \subset \mathbb{R}^l \), let \( f: V \to U \) be smooth. We let
\[
X_1, \ldots, X_k \text{ be coordinates on } U
\]
\[
y_1, \ldots, y_l \text{ be coordinates on } V
\]
If
\[
f = (f_1(y_1, \ldots, y_l), \ldots, f_k(y_1, \ldots, y_l))
\]
then
\[
df_y = \left( \frac{\partial f_i}{\partial y_j}(y) \right)
\]
and
\[
(df_y)^* = \left( \frac{\partial f_i}{\partial y_j}(y) \right)^T \text{ the transpose matrix}
\]
so
\[
f^* \, dx_i = \begin{pmatrix}
\frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_k}{\partial y_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_1}{\partial y_l} & \cdots & \frac{\partial f_k}{\partial y_l}
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
1 \\
0
\end{pmatrix} = \sum \frac{\partial f_i}{\partial y_j} \, dy_j
Now we call this
\[ df_i = \sum_{j=1}^{e} \frac{\partial f_i}{\partial y_j} dy_j \]

so using linearity and distributivity over \( \wedge \), we can write for
\[ \omega = \sum_I a_I \, dx_I \]
that
\[ f^* a_I (y) = a_I (f(y)) \]

\[ f^* \omega = \sum_I (f^* a_I) \, df_I \]

where if \( I = i_1, \ldots, i_p \), we have
\[ df_I = df_{i_1} \wedge \ldots \wedge df_{i_p} \]

This theorem looks great, but is not terribly useful. We give a special case.

Suppose \( k = e \) and \( \omega = dx_1 \wedge \ldots \wedge dx_k \) (the "volume form" on \( \mathbb{R}^k \)). Then
\[ f^* \omega = (\det df_y) \, \omega \]