Differentiation of forms

We have seen how to integrate forms. Further, we have seen that if $f$ is a diffeomorphism integration and pullback transform naturally into one another:

$$\int_X f^* \omega = \int_Y \omega, \text{ when } f : X \rightarrow Y.$$ 

We will now define a natural version of differentiation for forms on $\mathbb{R}^k$:

Recall: If $f$ is a 0-form, $df = \sum_{j} \frac{\partial f}{\partial x_j} \, dx_j$

is a 1-form.

Definition. If $\omega = \sum a_I \, dx_I$, then

$$d\omega = \sum da_I \wedge dx_I.$$
We note that

Theorem.

(1). If \( \omega \) is a \( p \)-form, \( d\omega \) is a \( p+1 \)-form.

(2). \( d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2 \)

(3). \( d(\omega \wedge \Theta) = (d\omega) \wedge \Theta + (-1)^p \omega \wedge d\Theta \)
   if \( \omega \) is a \( p \)-form.

(4). \( d(d\omega) = 0 \).

(5). \( d \) is the only operator so that \( df = \langle -\nabla f \rangle \)
for functions with these properties.

Proof. (1), (2) are obvious.

(3). Well, if \( \omega = \sum_{i} a_i \, dx_i \), \( \Theta = \sum_{j} b_j \, dx_j \)
then \( \omega \wedge \Theta = \sum_{i,j} a_i b_j \, dx_i \wedge dx_j \), so

\[
d(\omega \wedge \Theta) = \sum_{i,j} a_i b_j \, da_i \wedge dx_i \wedge dx_j + a_i \, db_j \wedge dx_i \wedge dx_j \]
\[
= \sum_{i,j} (da_i \wedge dx_i) \wedge b_j \, dx_j + \sum_{i,j} a_i \, dx_i \wedge db_j \wedge dx_j \]
\[
= d\omega \wedge \Theta + (-1)^p \omega \wedge d\Theta.
\]
To prove (4), we observe

\[ d\omega = \sum_I da_I \wedge dx_I \]

\[ = \sum_I \left( \sum_i \frac{\partial a_I}{\partial x_i} \right) dx_i \wedge dx_I \]

So

\[ d(d\omega) = \sum_{Ij} d \left( \frac{\partial a_I}{\partial x_i} \right) dx_i \wedge dx_i \wedge dx_I \]

\[ = \sum_{Ij} \frac{\partial^2 a_I}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I \]

But for each pair \((ij\), the corresponding pair \((ji\) also appears in the sum. Since \(\frac{\partial^2 a_I}{\partial x_j \partial x_i} = \frac{\partial^2 a_I}{\partial x_i \partial x_j}\) we cancel these two by two.
(5). Suppose we had another operator $D$ with (1)-(4) true and $Df = df$ for functions. Observe

$$D(dx_I) = D(dx_{I_1} \wedge \ldots \wedge dx_{I_p})$$

$$= \sum \pm dx_{I_1} \wedge \ldots \wedge Dx_k \wedge \ldots \wedge dx_{I_p}$$

But $dx_k = Dx_k$, so $D dx_k = DDx_k = 0$, so

$$D(dx_I) = 0.$$

We now let $\omega$ be any $p$-form,

$$D\omega = \sum_I \left[ D(a_I) \wedge dx_I + a_I \frac{D(dx_I)}{0} \right]$$

$$= \sum_I D(a_I) \wedge dx_I = \sum da_I \wedge dx_I = dw.$$
Corollary. Suppose \( g: V \to U \) is a diffeomorphism, and \( V \subseteq \mathbb{R}^k, U \subseteq \mathbb{R}^k \). Then for any \( \omega \) on \( U \),
\[
d(g^* \omega) = g^*(d\omega).
\]

Proof. It is easy to see
\[
D = (g^{-1})^* \circ d \circ g^*
\]
obeys the properties above \((2)-(4)\). We already know for functions that
\[
d(g^* f) = g^*(df)
\]
so \( D = d \), or
\[
\&(g^{-1})^* \circ d \circ g^* = d
\]
or
\[
d \circ g^* = g^* \circ d.
\]
We will use this transformation property to define $d$ for forms on manifolds.

For a form $\omega$ on $\Omega$, we let

$$d\omega = (\phi^{-1})^*d(\phi^*\omega)$$

Now if $g = \phi^{-1} \circ \psi$, we already know

$$g^*(d(\phi^*\omega)) = d(g^*(\phi^*\omega))$$

$$= d(\psi^*\omega)$$
\[(\psi^{-1})^* d (\psi^* \omega) = (\psi^{-1})^* g^* d(\varphi^* \omega) \]
\[= (g \circ \psi^{-1})^* d(\varphi^* \omega) \]
\[= (\varphi^{-1})^* d(\varphi^* \omega). \]

So (as expected) this doesn't depend on the choice of coordinates.

All of the previous properties hold for \(d\) on manifolds. So we can show

Theorem. Let \(g: \psi \to X\) be any smooth map of manifolds (which may have boundary). Then for any \(\omega\) on \(X\),
\[d(g^* \omega) = g^* (d\omega).\]
Observation: We know this is true for any 0-form \( f \). Further, for any \( df \),

\[ d(g^*df) = d(d g^*f) = 0 \]

and

\[ g^*(d(df)) = g^*(0) = 0. \]

Now suppose the theorem holds for \( \omega \) and \( \Theta \). It is easy to compute

\[ d(g^*(\omega \wedge \Theta)) = d(g^*\omega \wedge g^*\Theta) \]

\[ = d(g^*\omega) \wedge g^*\Theta + (-1)^p g^*\omega \wedge g^*d\Theta \]

\[ = g^*(d\omega) \wedge g^*\Theta + (-1)^p g^*\omega \wedge g^*d\Theta \]

\[ = g^*(d\omega \wedge \Theta + (-1)^p \omega \wedge d\Theta) \]

\[ = g^*(d(\omega \wedge \Theta)). \]

So it holds for \( \omega \wedge \Theta \) as well.
However, every $\omega = \sum a_i \, dx_i$ so since the theorem holds for the $a_i$ and $\frac{\partial}{\partial x_i}$, we're done.

Examples. In $\mathbb{R}^3$,

1. $\omega = f_1 \, dx_1 + f_2 \, dx_2 + f_3 \, dx_3$ is a vector field. Then

$$d\omega = df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3$$

$$= \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) \, dx_2 \wedge dx_3$$
$$+ \left( \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) \, dx_3 \wedge dx_1$$
$$+ \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \, dx_1 \wedge dx_2$$

$d\omega$ is $\text{curl} \, \omega$. 

\[ \omega = f_1 \, dx_2 \wedge dx_3 + f_2 \, dx_3 \wedge dx_1 + f_3 \, dx_1 \wedge dx_2. \]

\[ d\omega = \left( \frac{\partial f_1}{\partial x_4} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3. \]

\[ = \left( \text{div} \left( f_1, f_2, f_3 \right) \right) \, dx_1 \wedge ... \wedge dx_3. \]

\[ dw \text{ is } \text{div} \, \omega. \]

This means (satisfying, isn't it?) that \text{grad}, \text{curl}, and \text{div} are all really \text{d}.

And explains the facts

\[ \text{curl} \left( \text{grad} \, f \right) = 0 \]

\[ \text{div} \left( \text{curl} \, f \right) = 0 \]

in a natural way.