Integration and Mappings

We have previously proved that when $f: X \to Y$ is a diffeomorphism, we have

$$\int_X f^* \omega = \pm \int_Y \omega$$

for any k-form on Y (when X, Y are compact, oriented), where the sign depends on whether f preserves or reverses orientation. We now generalize:

Theorem. If $f: X \to Y$ is any smooth map between compact oriented manifolds of dimension k, then for any k-form ω on Y,

$$\int_X f^* \omega = \deg(f) \int_Y \omega.$$

We start with a special case:

Theorem. If $X = \partial W$ and $f: X \to Y$ extends to $F: W \to Y$ then

$$\int_X f^*\omega = 0$$

for any ω on Y, with $\dim \omega = \dim Y$.

Proof. We compute

$$\int_X f^*\omega = \int_{\partial W} F^*\omega = \int_W d\,F^*\omega$$

$$= \int_W F^* d\omega$$

But $d\omega$ is a $(K+1)$-form on the K-manifold Y, so $d\omega = 0$, completing the proof.

Corollary. If $f_0, f_1 : X \to Y$ are homotopic maps of the K-manifolds X and Y, then for every K-form ω on Y,

$$\int_X f_0^* \omega = \int_X f_1^* \omega .$$

Proof. Let the homotopy be $F : X \times I \to Y$. We know $\partial(X \times I) = X_1 - X_0$, so by the theorem

$$0 = \int_{\partial(X \times I)} (\partial F)^* \omega = \int_{X_1} (\partial F^*)^* \omega - \int_{X_0} (\partial F)^* \omega ,$$

but on X_1, $\partial F = f_1$ and on X_0, $\partial F = f_0$. $\therefore$

We need one last lemma:

Lemma. Let y be a regular value of $f : X \to Y$. $\exists$ a neighborhood U of y so that

$$\int_X f^* \omega = \deg(f) \int_Y \omega$$

for every ω supported in U.

Proof. By the stack of records theorem, $\exists$ a bunch of disjoint $V_1, \dots, V_n \subset X$ so that $f: V_i \to U$ is a diffeomorphism for all i. Further,

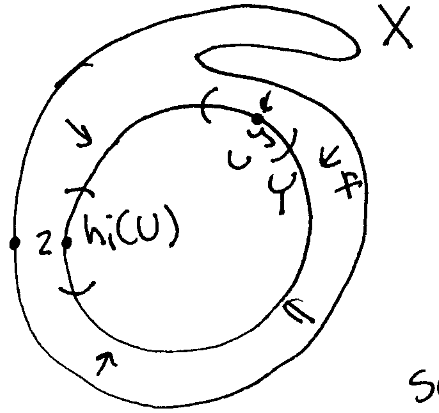
$$f^*\omega \text{ is supported in } \bigcup V_i.$$

So

$$\int_X f^*\omega = \sum_i \int_{V_i} f^*\omega = \sum_i \sigma_i \int_U \omega$$

where $\sigma_i = \pm 1$, depending on whether $f: V_i \to U$ preserves or reverses orientation. We already know $\sum_i \sigma_i = \deg f$. $\therefore$

We are now ready to prove the theorem!



Pick a regular value y and neighborhood U. By the isotopy lemma, $\forall z \in Y$ $\exists$ $h: Y \to Y$ homotopic* to I so that $h(y) = z$ and $h(U)$ is an open neighborhood of z, where h is a diffeo.

By compactness, ~~th~~ we can cover Y with some $h_1(U), \dots, h_n(U)$.

Use a partition of unity to write ω as a sum of forms, each supported on one of the $h_i(U)$.

Take any such ω.
Since $h \sim I$, $h \circ f \sim f$, so by Corollary

$$\int_X f^*\omega = \int_X (h \circ f)^*\omega = \int_X f^*(h^*\omega)$$

but $h^*\omega$ is supported in U, so by the Lemma

$$\int_X f^*(h^*\omega) = \deg(f) \int_Y h^*\omega$$

But h is a diffeomorphism of Y which is orientation <u>preserving</u> since it is $\sim$ to I, so

$$\deg(f) \int_Y h^*\omega = \deg(f) \int_Y \omega,$$

as claimed.