Integration and Mappings

We have previously proved that when \( f: X \to Y \) is a diffeomorphism, we have

\[
\int_X f^* \omega = \pm \int_Y \omega
\]

for any \( k \)-form on \( Y \) (when \( X, Y \) are compact, oriented), where the sign depends on whether \( f \) preserves or reverses orientation. We now generalize:

Theorem. If \( f: X \to Y \) is any smooth map between compact oriented manifolds of dimension \( k \), then for any \( k \)-form \( \omega \) on \( Y \),

\[
\int_X f^* \omega = \deg(f) \int_Y \omega.
\]
We start with a special case:

**Theorem.** If $X = 2W$ and $f: X \to Y$ extends to $F: W \to Y$ then

$$\int_X f^* w = 0$$

for any $w$ on $Y$, with $\dim w = \dim Y$.

**Proof.** We compute

$$\int_X f^* w = \int_{2W} F^* w = \int_W d F^* w$$

$$= \int_W F^* dw$$

But $dw$ is a $(k+1)$-form on the $k$-manifold $Y$, so $dw = 0$, completing the proof.
Corollary. If \( f_0, f_1 : X \to Y \) are homotopic maps of the \( k \)-manifolds \( X \) and \( Y \), then for every \( k \)-form \( \omega \) on \( Y \),

\[
\int_X f_0^* \omega = \int_X f_1^* \omega.
\]

Proof. Let the homotopy be \( F : X \times I \to Y \).

We know \( \frac{\partial \theta}{\partial t} = X_1 - X_0 \), so by the theorem

\[
\theta(\partial \theta) = \int_{\theta(X \times I)} \theta(\partial \theta)^* \omega = \int_{\theta(X_1 \times I)} (\partial \theta)^* \omega - \int_{\theta(X_0 \times I)} (\partial \theta)^* \omega,
\]

but on \( X_1 \), \( \partial \theta = f_1 \) and on \( X_0 \), \( \partial \theta = f_0 \). 

We need one last lemma:

Lemma. Let \( y \) be a regular value of \( f : X \to Y \). \exists a neighborhood \( U \) of \( y \) so that

\[
\int_{\partial X} f^* \omega = \text{deg}(f) \int_X \omega
\]

for every \( \omega \) supported in \( U \).
Proof. By the stack of records theorem, \( \exists \) a bunch of disjoint \( V_1, \ldots, V_n \subset X \) so that \( f: V_i \to U \) is a diffeomorphism for all \( i \). Further, \( f^* \omega \) is supported in \( U V_i \).

So
\[
\int_X f^* \omega = \sum_i \int_{V_i} f^* \omega = \sum_i \sigma_i \int_U \omega
\]
where \( \sigma_i = \pm 1 \), depending on whether \( f: V_i \to U \) preserves or reverses orientation.

We already know \( \sum_i \sigma_i = \deg f \). \( \therefore \)

We are now ready to prove the theorem!

\( \odot \) Pick a regular value \( y \) and neighborhood \( U \).

By the isotopy lemma, \( \forall z \in Y \)
\( \exists \ h: Y \to Y \) homotopic to \( I \)
so that \( h(y) = z \) and \( h(U) \) is an open neighborhood of \( z \), where \( h \) is a diffeo.
By compactness, we can cover \( Y \) with some \( h_1(U), \ldots, h_n(U) \).

Use a partition of unity to write \( w \) as a sum of forms \( h_i(U) \), supported on each one of

Take any such \( w \).

Since \( h \circ I \), \( h \circ f \) is a diffeomorphism of \( Y \) which is orientation preserving since it is \( \sim \) to \( I \), so

\[
\int_X f^* w = \int_X (h \circ f)^* w = \int_X \left( \sum h_i^* w \right)
\]

but \( h_i^* w \) is supported in \( U \), so by the

\[
\int_X f^*(h_i^* w) = \deg(f) \int_Y h_i^* w
\]

But \( h \) is a diffeomorphism of \( Y \) which is orientation preserving since it is \( \sim \) to \( I \), so

\[
\deg(f) \int_Y h_i^* w = \deg(f) \int_Y w,
\]

as claimed.