Math 4250 - Review

We are going to start gently by recalling some facts about linear algebra and calculus.

Definitions.

A vector \( \vec{v} \in \mathbb{R}^n \) is a list \( \vec{v} = (v_1, \ldots, v_n) \).

\[ \vec{v} + \vec{w} = (v_1 + w_1, \ldots, v_n + w_n) \]

\[ k\vec{v} = (kv_1, \ldots, kv_n) \]

\[ \langle \vec{v}, \vec{w} \rangle = \text{ or } \vec{v} \cdot \vec{w} = \sum v_i w_i \]

The length of \( \vec{v} \) is given by \( ||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle} \).

A set of vectors \( \vec{v}_1, \ldots, \vec{v}_n \) forms a basis for \( \mathbb{R}^n \) if and only if every \( \vec{w} \in \mathbb{R}^n \) can be written uniquely as \( \vec{w} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n \) for some \( c_1, \ldots, c_n \).
If we write

\[
A = \begin{bmatrix}
\hat{V}_1 & \vdots & \hat{V}_n \\
\downarrow & \cdots & \downarrow \\
\vdots & \vdots & \vdots \\
\downarrow & \cdots & \downarrow \\
\hat{V}_n & \cdots & \hat{V}_n
\end{bmatrix}
\]

then \( c_1 \hat{V}_1 + \ldots + c_n \hat{V}_n = \hat{\omega} \iff \vec{c} \) is the solution to the matrix equation

\[
A \vec{c} = \hat{\omega}
\]

since

\[
\begin{bmatrix}
\hat{V}_1 & \cdots & \hat{V}_n \\
\vdots & \vdots & \vdots \\
\hat{V}_n & \cdots & \hat{V}_n
\end{bmatrix}
\begin{bmatrix}
c_1 \\
\vdots \\
c_n
\end{bmatrix}
= \begin{bmatrix}
c_1 \hat{V}_{11} + c_2 \hat{V}_{21} + \ldots + c_n \hat{V}_{n1} \\
\vdots \\
c_1 \hat{V}_{1n} + \ldots + c_n \hat{V}_{nn}
\end{bmatrix}
= c_1 \hat{V}_1 + \ldots + c_n \hat{V}_n.
\]

Prop. \( \hat{V}_1, \ldots, \hat{V}_n \) is a basis \( \iff \) \( A \) is an invertible matrix \( \iff \) \( \det A \neq 0 \).
Definition. An nxn matrix is orthogonal if

\[ A A^T = I = A^T A \] (or \[ A^T = A^{-1} \])

Lemma. An nxn matrix \( A \) is orthogonal iff the column vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) are unit length and pairwise orthogonal.

Proof.

\[
A^T A = \begin{bmatrix}
\mathbf{v}_1^T & \cdots & \mathbf{v}_n^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_1 & \cdots & \mathbf{v}_n
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\langle \mathbf{v}_i, \mathbf{v}_j \rangle
\end{bmatrix}
\]

This last matrix of dot products (called the Gramian) is the identity.
iff the $\vec{v}_i$ are unit length (so $\langle \vec{v}_i, \vec{v}_i \rangle = 1$) and pairwise orthogonal (so $\langle \vec{v}_i, \vec{v}_j \rangle = 0$).

Note that if $A$ is an orthogonal matrix and we want to write a vector $\vec{w}$ as a linear combination $c_1\vec{v}_1 + \cdots + c_n\vec{v}$ of the columns of $A$, it's easy!

$$A \hat{c} = \hat{w}$$

so

$$\hat{c} = A^{-1} \hat{w}$$

so

$$\hat{c} = A^T \hat{w} = \begin{bmatrix} \leftarrow \vec{v}_1 \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} \langle \vec{v}_1, \hat{w} \rangle \\ \vdots \\ \langle \vec{v}_n, \hat{w} \rangle \end{bmatrix}.$$
Explicitly, if $\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_n$ are orthonormal,

$$\hat{\mathbf{w}} = \langle \hat{\mathbf{w}}, \hat{\mathbf{v}}_1 \rangle \hat{\mathbf{v}}_1 + \ldots + \langle \hat{\mathbf{w}}, \hat{\mathbf{v}}_n \rangle \hat{\mathbf{v}}_n.$$ 

Example. Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is orthogonal. We know

$$a_{11}^2 + a_{21}^2 = 1 \quad a_{12}^2 + a_{22}^2 = 1.$$ 

$$a_{11} a_{12} + a_{21} a_{22} = 0.$$ 

Solving the last equation,

$$\frac{a_{11}}{a_{21}} = -\frac{a_{22}}{a_{12}}$$

which means that there is some $\lambda$ so that

$$A = \begin{bmatrix} a_{11} & -\lambda a_{21} \\ a_{21} & \lambda a_{11} \end{bmatrix}$$
Further, this means

\[ a_{11}^2 + a_{21}^2 = 1 \]
\[ \lambda a_{11}^2 + \lambda a_{21}^2 = 1 \]

so

\[ \lambda^2 = 1 \quad \text{and} \quad \lambda = \pm 1. \]

This means there are two cases:

\[
A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}
\]

Since \( a^2 + b^2 = 1 \), there is some angle \( \Theta \) so that \( a = \cos \Theta, \ b = \sin \Theta \). Then

\[
\begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix}
\]
maps

\[
(0,1) \rightarrow (1,0) \rightarrow (-\sin \Theta, \cos \Theta) \rightarrow (\cos \Theta, \sin \Theta)
\]
and we see this is a rotation by angle $\theta$.

The other matrix,

$$
\begin{bmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta 
\end{bmatrix}
$$

turns out to be a reflection over the line $(\cos \frac{\theta}{2}, \sin \frac{\theta}{2})$.

We won't prove this in class, as it's a homework exercise, but the picture should convince you a bit.
Definition. The orthogonal group $O(n)$ consists of all $n \times n$ orthogonal matrices. The special orthogonal group $SO(n)$ is the subgroup of $O(n)$ of matrices of determinant $+1$.

We can show

Proposition. Every matrix $A \in SO(3)$ is a rotation around some axis $\hat{\mathbf{v}}$.

(Proof. Homework, but the hint is that $\hat{\mathbf{v}}$ is the eigenvector of $A$ with eigenvalue $1$.)
If you remember the cross product for vectors in \( \mathbb{R}^3 \),

\[
\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)
\]

It has the properties:

\[
\langle \mathbf{v}, \mathbf{v} \times \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \times \mathbf{w} \rangle = 0
\]

\[
\| \mathbf{v} \times \mathbf{w} \| = \| \mathbf{v} \| \| \mathbf{w} \| \sin \theta
\]

\[
\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}
\]

Proposition. The columns of a matrix \( A \) in \( \text{SO}(3) \) are in the form

\[
A = \begin{bmatrix}
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_4 \times \mathbf{v}_2 \\
\mathbf{v}_1 \times \mathbf{v}_2 & \mathbf{v}_2 & \mathbf{v}_4
\end{bmatrix}
\]

Proof. (Again, homework!)
We close by recalling some useful identities:

\[
\langle \hat{a}, \hat{b} \times \hat{c} \rangle = \det \begin{bmatrix} \hat{a} & \hat{b} & \hat{c} \\ \hat{i} & \hat{i} & \hat{i} \end{bmatrix}
\]

is called the "triple product."

\[
\langle \hat{a}, \hat{b} \times \hat{c} \rangle = \langle \hat{b}, \hat{c} \times \hat{a} \rangle = \langle \hat{c}, \hat{a} \times \hat{b} \rangle
\]

(cyclic permutation)

\[
\hat{a} \times (\hat{b} \times \hat{c}) = \hat{b} \langle \hat{a}, \hat{c} \rangle - \hat{c} \langle \hat{a}, \hat{b} \rangle
\]

(\text{truck} = \begin{bmatrix} \text{back} & \text{cab} \end{bmatrix})