3.1. Finding Roots of Functions.

We are interested in solving equations of the form $f(x) = 0$ for $x$.

\[ \text{Graph of a function with zeros at distinct points.} \]

Idea 1. Given an interval $[a, b]$ where \( \text{sign } f(a) \neq \text{sign } f(b) \), we compute $f(c)$ where $c = \frac{a+b}{2}$ and restrict our attention to either $[a, c]$ or $[c, b]$ depending on \( \text{sign } f(c) \).

This is called the bisection method. \&

<Mathematica demonstration>
After \( n \) steps, the error in the computed position of the root is at most \( \frac{b-a}{2^n+1} \).

**Definition.** If \( \{x_n\} \rightarrow x \), then the sequence has **linear convergence** if \( \exists C \in (0,1) \) so that

\[
|x_{n+1} - x| \leq C |x_n - x|
\]

**Lemma.** If \( \{x_n\} \rightarrow x \) linearly with constant \( C \), then \( |x_{n+1} - x| \leq AC^n \), where \( A = |f'(x_0)| \).

**Question.** Does the bisection method converge linearly? (We take the sequence to be the sequence of midpoints, and \( x \) to be whatever root the method converges to.)
Consider

On step 1, we replace our initial guess of $a + \frac{atb}{2}$ by $\frac{a + atb}{2} = \frac{3}{4}a + \frac{1}{4}b = x_1$.
But $x_0$ was actually closer to the root than $x_1$!

We see that bisection is not guaranteed to improve at each step.

On the other hand, bisection does give the conclusion of the lemma, which is most equally useful in practice.
We can improve the bisection method by changing our choice of midpoint.

The "false position" method guesses that the function is well approximated by the secant line through \((a,f(a))\) and \((b,f(b))\) and chooses the guess for the new zero accordingly.

This can have linear (and even superlinear) convergence if the details are handled right... we will return to this method soon!
What if we can calculate a derivative of our function?

Estimating where the tangent line crosses the x-axis is the basis for Newton's Method.

Doing the algebra establishes that

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

<newton_method.nb demonstration>
We saw that the number of correct digits increases exponentially. To be more precise, we will prove:

**Definition.** We say $x_{n+1} \rightarrow x$ *quadratically* if $|x_{n+1} - x| \leq c |x_n - x|^2$ for some $c$.

Observe that if $x_n$ has $K$ correct decimal digits, then $|x_n - x| < 10^{-K}$, so

$|x_{n+1} - x| < c(10^{-K})^2 = c \cdot 10^{-2K}$, and the number of correct digits has approximately doubled (depending on $c$).

**Newton’s Method Theorem.** If $f, f', f''$ are continuous in a neighborhood of a root $r$ of $f$, and $f'(r) \neq 0$, there is a neighborhood $N_S$ of $r$ of radius $S$ so that if $x_0 \in N_S$ then all $x_n \in N_S$ and

$$|r - x_{n+1}| \leq c(S) |r - x_n|^2$$

for some $c$ depending on $f$ and $S$ (given below).
Proof. Let $e_n = r - x_n$. We know

$$e_{n+1} = r - x_{n+1} = r - (x_n - \frac{f(x_n)}{f'(x_n)})$$

$$= (r - x_n) + \frac{f(x_n)}{f'(x_n)}$$

$$= e_n + \frac{f(x_n)}{f'(x_n)}$$

$$= \frac{e_n f'(x_n) + f(x_n)}{f'(x_n)}.$$ 

Now let's Taylor expand $f$ around $x_n$. We know

$$0 = f(r) = f(x_n + e_n) = f(x_n) + e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n),$$

where $\xi_n \in [x_n, r]$. This means

$$e_{n+1} = -\frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} e_n^2,$$

which is almost what we want.
Observe that we can define a function
\[ c(\delta) = \frac{1}{\alpha} \frac{\max_{N_\delta} |f''(x)|}{\min_{N_\delta} |f'(x)|} \]
which is finite for small enough \( \delta \). In fact, we can choose \( \delta \) small enough that
\[ \delta c(\delta) < 1 \]
since as \( \delta \to 0 \), \( c(\delta) \to \frac{f''(r)}{f'(r)} \). Now all we have to observe is that if \( x_n \in N_\delta \),
\[ \left| \frac{1}{\alpha} \frac{f''(x_n)}{f'(x_n)} \right| \leq c(\delta) \]
since, \( x_n, \xi_n \) are in \( N_\delta \). In this case
\[ |e_{n+1}| \leq c(\delta) e_n^2 \leq \delta c(\delta) e_n < e_n, \]
so \( x_{n+1} \) is in \( N_\delta \) as well. So if we choose \( x_0 \) in \( N_\delta \), all subsequent \( x_n \) are also in \( N_\delta \).
The last thing to show is that \( \exists n_3 \to r \). But we know

\[ |e_n| \leq \delta c(\delta) |e_{n-1}| < (\delta c(\delta))^2 |e_{n-2}| < \cdots < (\delta c(\delta))^n e_0. \]

and since \( \delta c(\delta) < 1 \), this means

\[ \exists n_3 \to 0. \quad \Box \]

Notice that our test function

\[ f(x) = (x-1)^3 \]

has

\[ f'(x) = 3(x-1)^2 \]

and \( f''(1) = 0 \), so the quadratic convergence theorem doesn’t hold

(and we only got linear convergence!)
As with the bisection method, if $f(x)$ has multiple roots, then there's no guarantee which one Newton will converge to.

In fact, predicting this can be quite hard!