Errors in Polynomial Interpolation

Suppose we want to approximate a (nice) function $f(x)$ with a polynomial $p(x)$ on a fixed interval $[a, b]$.

We choose nodes $x_0, \ldots, x_n$ in $[a, b]$, and compute. Clearly $f(x) = p(x)$ at $x_0, \ldots, x_n$. Let

$$E(n) = \max_{x \in [a, b]} |f(x) - p(x)|.$$ 

It is natural to expect $E(n) \to 0$ as $n \to \infty$. In fact, for evenly spaced nodes, it is often the case that $E(n) \to \infty$ as $n \to \infty!$
Case study: \( f(x) = \frac{1}{1+x^2} \) on \([-1,1]\).

\(<\text{chebyshev-nodes. nb}>\)

We can give a few theorems about polynomial interpolation to shed some light on the matter.

Theorem. If \( p \) is the degree (at most) \( n \) polynomial interpolating \( f \) at \( n+1 \) distinct nodes \( x_0, \ldots, x_n \) in \([a,b]\) and \( f^{(n+1)} \) is continuous, then for each \( x \in [a,b] \) there is some \( \xi \in (a,b) \) so

\[
f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x-x_i).
\]
Proof. If $x$ is a node, we're done. If not, let us fix $x$ and set

$$\omega(t) = \prod_{i=0}^{n} (t-x_i)$$

$$c = \frac{f(x) - p(x)}{\omega(x)} \quad \text{← this is a constant}$$

$$\Phi(t) = f(t) - p(t) - c \omega(t).$$

(Since $x$ is not a node, $\omega(x) \neq 0$, so $c$ exists.)

Now $\Phi(t) = 0$ at all the nodes $x_0, \ldots, x_n$ and $x$. We recall Rolle's theorem:

Theorem. Between any two roots of $f$ there is a root of $f'$. 
The interval $[a,b]$ now contains

- $n+2$ roots of $\Phi$
- $n+1$ roots of $\Phi'$
- $n$ roots of $\Phi''$
- \vdots
- 1 root of $\Phi^{(n+1)}$

At this point (call it $\xi$), we have

$$0 = \Phi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - p^{(n+1)}(\xi) + c \omega^{(n+1)}(\xi).$$

Now $p$ is a polynomial of degree $n$, so $p^{(n+1)}(x) \equiv 0$. Now $\omega(t)$ is a polynomial of degree $n+1$ with leading term $t^{n+1}$, so $\omega^{(n+1)}(t) \equiv (n+1)!$. So

$$0 = f^{(n+1)}(\xi) + c (n+1)!.$$  

Substituting in the definition of $c$... □
Of course, this is not very helpful unless we can estimate $\prod (x-x_i)$. Lemma. If $x_i=a+ih$, $i=0,\ldots,n$, $h=b-a/n$, then for any $x \in [a,b]$, 

$$\prod_{i=0}^{n} |x-x_i| \leq \frac{1}{4} h^{n+1} n!$$

Proof. If $x \in [x_j,x_{j+1}]$ we can show (homework) that 

$$|x-x_j||x-x_{j+1}| \leq \frac{h^2}{4}.$$ 

So 

$$\prod_{i=0}^{n} |x-x_i| \leq \left( \prod_{i=0}^{j-1} (x-x_i) \right) \frac{h^2}{4} \left( \prod_{i=j+2}^{n} (x_i-x) \right)$$

Now for $x_i \in [x_0,x_{j+1}]$, we have $x_{j+1} > x > x_j > x_i$, so $x-x_i < x_{j+1}-x_i$. 
Similarly for \( x_i \in [x_{j+1}, x_n] \) we have

\[ x_i > x_{j+1} > \ldots > x_j, \text{ so } x_i - x > x_i - x_j. \]

So we have

\[
\leq \left( \prod_{i=0}^{j-1} x_{j+1} - x_i \right) \frac{h^2}{4} \left( \prod_{i=j+2}^{n} x_i - x_j \right)
\]

But \( x_{j+1} - x_i = (j+1-i)h \) and \( x_i - x_j = (i-j)h \). So we really have

\[
\leq h^j \cdot \frac{h^2}{4} \cdot h^{n-(j+2)+1} \cdot \prod_{i=0}^{j-1} (j-i+1) \prod_{i=j+2}^{n} i-j
\]

\[
\leq h^{2+j+n-j-2+1} \frac{1}{4} (j+1)! (n-j)!
\]

\[
\leq \frac{1}{4} h^{n+2} n!
\]

where we use \((j+1)! (n-j)! \leq n!\) for \(j = 0, \ldots, n\). For \(j\) "in the middle" this is a gross overestimate but for \(j=0\) we
can do no better. □

We can now combine these to get

Theorem. If $f^{(n+1)}$ is continuous on $[a,b]$ and bounded by $M$ on $[a,b]$, then

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)} M h^{n+1}.$$ 

We can see from this that the key question is the growth of $M$ as a function on $n$. 
Interpolation Errors and Divided Differences.

Last time we saw that in general for a polynomial we had

\[ f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x-x_i) \]

Our mathematica examples show us that this error bound can blow up, along with the actual error.

<gridpoints_and_interpolation.nb>

Theorem. The Chebyshev nodes minimize

\[ \max_{x \in [a,b]} \left| \prod_{i=0}^{n} (x-x_i) \right| \]

over all choices of \( x_i \).

In fact, for Chebyshev nodes we have

\[ |f(x) - P_n(x)| \leq \frac{1}{2^n(n+1)!} M, \text{ where } M \text{ bounds } f^{(n+1)}(x) \]
(note that this last bound only works on \([-1,1]\)).

Now we show

Theorem. If \( p \) interpolates \( f \) on \( x_0, \ldots, x_n \) then for any \( x \) which is not a node,

\[
f(x) - p(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{i=0}^{n} (x-x_i).
\]

Proof. We know that if \( q \) interpolates \( f \) at \( x_0, \ldots, x_n \) and \( \forall t \) then

\[
q(x) = p(x) + f[x_0, \ldots, x_n, t] \prod_{i=0}^{n} (x-x_i)
\]

but at \( t=x \), \( q=f \), so we have

\[
f(x) - p(x) = f[x_0, \ldots, x_n, x] \prod_{i=0}^{n} (x-x_i). \quad \square
\]
Now this is weird and interesting because it suggests a connection between divided differences and derivatives:

Theorem. If $f^{(n)}$ is continuous on $[a,b]$, for any (distinct) $x_0, \ldots, x_n$ in $[a,b]$ there is some $\xi$ in $(a,b)$ so

$$f[x_0, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi).$$

Proof. Just combine these formulae for $f(x) - p(x)$. □

Corollary. If $f$ is a polynomial of degree $n$, then $f[x_0, \ldots, xi] = 0$ for $i > n$. 
Example. Are the data points

\[
\begin{array}{ccccccc}
  x & 1 & -2 & 0 & 3 & -1 & 7 \\
  y & -2 & -56 & -2 & 4 & -16 & 376 \\
\end{array}
\]

formed by sampling a cubic?

(compute divided diffs.)