Application: Geometric Derivatives

Suppose we have a space curve \( \mathbf{r}(s) \)

which we approximate by a polygonal line

How should we write down the tangent vector of the polygonal line at a vertex? How should we write the curvature or torsion of the poly line?
We start with a brief review of the differential geometry of curves.

Suppose $\hat{r}(s)$ is parametrized so that $|\hat{r}'(s)| = 1$. Then we say $\hat{r}$ is parametrized by arclength, since the length of the portion of $r$ between $s_0$ and $s_1$ is

$$\int_{s_0}^{s_1} |\hat{r}'(s)| \, ds = s_1 - s_0.$$ 

In this case, we can write

$$\hat{r}'(s) = T(s)$$

the unit tangent vector.
Now the rate of change of $T(s)$ measures how fast the curve is turning, so we let

$$|T'(s)| = k$$

the curvature of $r$

and define the normal vector of $r(s)$ by

$$T'(s) = k N(s).$$

Now the derivative of $N(s)$ must be perpendicular to $N(s)$ since

$$\langle N(s), N(s) \rangle = 1$$

so

$$\frac{d}{ds} \langle N(s), N(s) \rangle = 2 \langle N'(s), N(s) \rangle = 0.$$

Thus

$$N'(s) = a(s) T(s) + b(s) B(s),$$
where \( B(s) = T(s) \times N(s) \) is the third vector in the orthogonal system of coordinates.

Now we can compute

\[
0 = \frac{d}{ds} \left( T(s), N(s) \right) \\
= \left( T'(s), N(s) \right) + \left( T(s), N'(s) \right) \\
= \left( k(s)N(s), N(s) \right) + a(s) \\
= k(s) + a(s).
\]

So \( a(s) = -k(s) \).
On the other hand, \( b(s) \) measures the rate at which \( N(s) \) is spinning around \( T(s) \) so we call it torsion \( \gamma(s) \). Thus, to recap

\[
T'(s) = kN(s)
\]
\[
N'(s) = -kT(s) + \gamma B(s)
\]

and we can compute

\[
0 = \frac{d}{ds} \langle B(s), N(s) \rangle
\]
\[
= \langle B'(s), N(s) \rangle + \langle B(s), -kT(s) + \gamma B(s) \rangle
\]
\[
= \langle B'(s), N(s) \rangle + \gamma \beta
\]

while

\[
0 = \frac{d}{ds} \langle B(s), T(s) \rangle
\]
\[
= \langle B'(s), T(s) \rangle + \langle B(s), T'(s) \rangle
\]
\[
= \langle B'(s), T(s) \rangle + 0.
\]
so our last equation becomes

\[ B'(s) = -\gamma N(s). \]

It is very important to have estimates.
As you can see, curvature and torsion determine everything about how the frame \( T(s), N(s), B(s) \) evolves with \( s \) as \( s \) changes.

How do we define \( T(s), N(s), B(s), k(s) \) and \( \gamma(s) \) for polylines? Assume

and \( c = |\mathbf{c}|, d = |\mathbf{d}| \) and so forth.
We now report some recent (2005) results of Langer, Beljaev, Seidel on this problem:

First, given

\[ \hat{d} \quad \hat{e} \quad \hat{t} \]

How should we combine \( \hat{d} \) and \( \hat{e} \) to get an approximation to \( T \) (called \( \hat{T} \))? It's natural to assume we should take

\[ \frac{\hat{d} + \hat{e}}{2} \quad \text{(weight by length)} \]

\[ \frac{\hat{d}}{\hat{d} + \hat{e}} \quad \text{(weight equally)} \]

and normalize them.
In fact:

**Theorem [LBS, 2005]**

The only linear combination of $\hat{d}$ and $\hat{e}$ which yields a second order approximation to $T$ is given by

$$\hat{T} = \frac{de}{d+e} \left( \frac{\hat{d}}{d^2} + \frac{\hat{e}}{e^2} \right).$$

This is the tangent to the circle through $P_2, P_0, P_1$.

The proof is a clever and involved Taylor series calculation. The intuition, though, is simple: as $P_1$ moves away from $P_0$ along $r(s)$, the secant $d$ gets farther from $T$ and should have less influence on $\hat{T}$. 
How should we compute $KN$? The first idea is to try to approximate the difference between successive tangents.

Theorem [LBS 2005]

$$\hat{KN} := \frac{2}{d+e} \left( \frac{\hat{e}}{e} - \frac{\hat{d}}{d} \right)$$

Is a linear approximation to $KN$ and a quadratic approximation if $d = e$. Further if $\varphi$ is the turning angle from $d + e$,

$$\hat{K} := \left| \frac{2}{d+e} \left( \frac{\hat{e}}{e} - \frac{\hat{d}}{d} \right) \right|$$

or

$$\hat{K} := \frac{2\varphi}{d+e}$$

have the same asymptotic error as $\hat{KN}$. 
What about torsion? Well, we know

\[ B'(s) = \gamma N(s). \]

And given three points we can estimate \( B \) by the normal of the plane through those points.

\[ b_0 = \frac{dx \, e}{ldx} \]

Continuing, we have

\[ b_1 = \frac{exf}{lxef}, \quad b_{-1} = \frac{cx}{|cx|} \]

We can then define change vectors

\[ \hat{\eta}_c = \langle b_1 \times b_0, \hat{E} \rangle \]

\[ \hat{\eta}_d = \langle b_{-1} \times b_0, \hat{T} \rangle \]

These measure the approximate norm of \( B' \) since we know if \( B_0 \) is heading in the \( N \) direction as it swings to \( B_1 \), then \( B_0 \times B_1 \) is parallel to \( T \).
We then have

Theorem [LBS 2005]

A linear approximation to torsion is given by

\[ \hat{\gamma}_d := \frac{3\hat{\eta}_d}{c+d+e} \]

and

\[ \hat{\gamma}_e := \frac{3\eta_e}{d+e+f}. \]

We can get a better estimate for torsion in terms of these quantities by (essentially) Richardson extrapolation, leading to a 5-point quadratic approximation for \( \gamma \).

**Step 4.2:** Combine \( \hat{\gamma}_d \) and \( \hat{\gamma}_e \) to eliminate leading error terms:

\[ \hat{\gamma} \approx \frac{a + 2b + c + d + e + f}{c + d + e + f}. \]