Boundary Value Problems

Consider the typical problem

$$x''(t) = f(t, x, x'), \quad a < t < b$$

with boundary conditions at a and b given by the matrix equation

$$A \begin{bmatrix} x(a) \\ x'(a) \end{bmatrix} + B \begin{bmatrix} x(b) \\ x'(b) \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

That is, we know two linear equations that $x(a), x'(a), x(b), x'(b)$ satisfy.

Typical cases are

$$x(a) = \gamma_1, \quad x(b) = \gamma_2$$

$$x(a) = \gamma_1, \quad x'(a) = \gamma_2$$

$$x'(a) = \gamma_1, \quad x'(b) = \gamma_2$$

Unlike the initial value problem, where existence and uniqueness of solutions are standard, the theory for such problems is weird.

Example.

$$x''(t) = -\lambda x(t), \qquad 0 < x < 1$$

$$x(0) = x(1) = 0.$$

The general solution to $x''(t) = -\lambda x(t)$ is

$$x(t) = A\cos(\sqrt{\lambda}\,t) + B\sin(\sqrt{\lambda}\,t)$$

If we throw in $x(0) = 0$, we see that

$$x(t) = B\sin(\sqrt{\lambda}\,t).$$

Now there are two cases.

a) $\sin(\sqrt{\lambda}) = 0$. Any B works. There are ∞'ly many solutions.

b) $\sin(\sqrt{\lambda}) \neq 0$. The only solution is $B = 0$, so $x(t) \equiv 0$.

Example

$$x''(t) = -\lambda x(t) + g(t)$$
$$x(0) = x(1) = 0.$$

If $\sin(\sqrt{\lambda}) \neq 0$, it turns out that this has a unique solution. Otherwise, this has a solution if and only if

if $\lambda = \pi^2$

$$\int_0^1 g(t) \sin(\pi t)\, dt = 0$$

In this case,

$$\cancel{g}\ x(t) = C \sin(\pi t) + \frac{1}{\pi} \int_0^t g(s) \sin(\pi(t-s))\, ds$$

(and similarly for other λ values).

If we have a <u>linear</u> problem

$$x''(t) = p(t)\,x'(t) + q(t)\,x(t) + g(t)$$

$$A\begin{bmatrix} x(a) \\ x'(a) \end{bmatrix} + B\begin{bmatrix} x(b) \\ x'(b) \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

then we say it is <u>homogenous</u> if $g(t) \equiv 0$, $\gamma_1, \gamma_2 = 0$.

<u>Theorem</u>. This problem has a unique solution for each $\{g(t), \gamma_1, \gamma_2\}$ $\Leftrightarrow$ the homogenous problem has <u>only</u> the trivial solution.

When might this happen? It turns out to be the case that this is true if we can write the boundary conditions in <u>separated</u> form

$$a_0 x(a) - a_1 x'(a) = \gamma_1$$
$$b_0 x(b) - b_1 x'(b) = \gamma_2$$

and we have

$$q(t) > 0,$$
$$a_0 a_1 \geq 0 \qquad b_0 b_1 \geq 0$$
$$|a_0| + |a_1| \neq 0, \ |b_1| + |b_0| \neq 0, \ |a_0| + |b_0| \neq 0.$$

Then we know the linear system has a unique solution.

What about the nonlinear case? Atkinson (p 435) describes the theory as "complicated." If boundary conditions are separated, and

$$|f(t, u_1, v) - f(t, u_2, v)| < K |u_1 - u_2|$$
$$|f(t, u, v_1) - f(t, u, v_2)| < K |v_1 - v_2|$$

and

$$\frac{\partial f}{\partial u} > 0, \quad \left| \frac{\partial f}{\partial v} \right| \leq M$$

and (as before)

$$a_0 a_1 \geq 0, \quad b_0 b_1 \geq 0$$

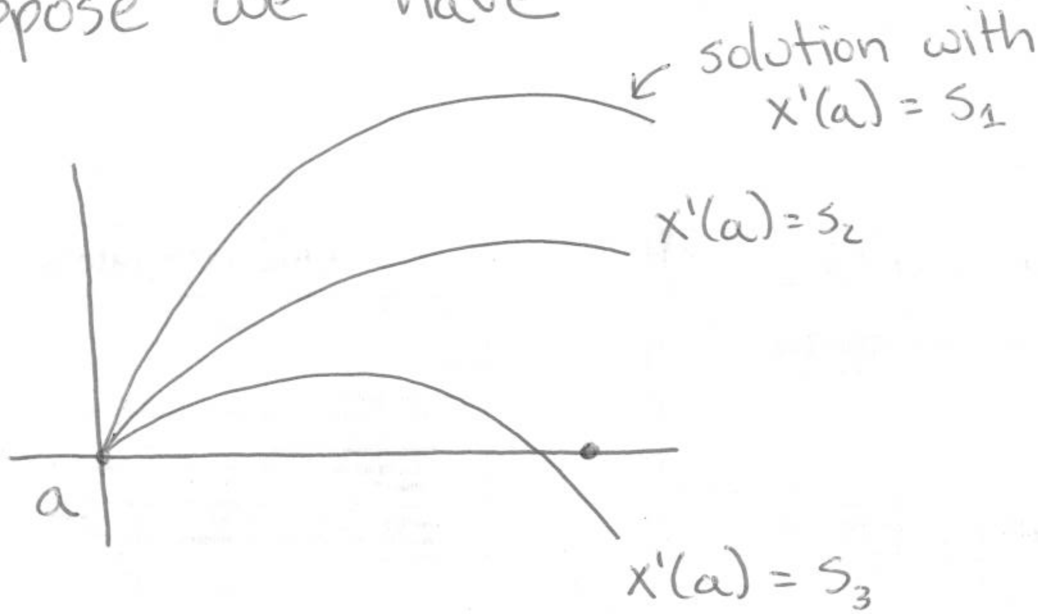
$$|a_0| + |a_1| \neq 0, \quad |b_0| + |b_1| \neq 0, \quad |a_0| + |b_0| \neq 0$$

then

<u>Theorem</u>. Under these conditions, the boundary value problem has a unique solution.

How can we find such a solution? The basic idea is as follows.

Suppose we have



We want to view

$$x^*(b) = \varphi(s)$$

where x is the solution with $x(a) = 0$ and $x'(a) = s$. This is a nonlinear eqn for s and we solve it by using an initial value method for $x(t)$.

Here are the details.

Suppose we have

$$x'' = f(t, x, x')$$

$$x(a) = a_1 s - c_1 \gamma_1$$

$$x'(a) = a_0 s - c_0 \gamma_1$$

where $a_1 c_0 - a_0 c_1 = 1$. Then we have (if $x(t;s)$ is a solution of this IVP),

~~system~~

$$a_0 x(a;s) - a_1 x(a;s) =$$

$$a_0(a_1 s - c_1\gamma_1) - a_1(a_0 s - c_0 \gamma_1)$$

$$= (a_0 c_1 - a_1 c_0)\gamma_1 = \gamma_1$$

for any s. We then have

$$\varphi(s) := b_0 x(b;s) + b_1 x'(b;s) - \gamma_2.$$

We are looking for s so that

we have ~~φ~~ $\varphi(s_*) = 0$. Then the solution to that IVP also solves our boundary value problem!

Next time: How to solve $\varphi(s) = 0$, demonstrations.