Condition Number for Least Squares.

It is an interesting feature of least squares that the difficulty of solving a given problem may depend on the rhs $b$ and not just on $A$.

We can still define a "condition number" $\kappa_2(A)$.

**Definition.** If $A$ is $m \times n$, $m > n$, we define

$$\kappa_2(A) := \frac{\sigma_1(A)}{\sigma_n(A)},$$

the ratio of largest and smallest condition numbers.
Theorem. Suppose $A$ is $m \times n$ with $m \geq n$ and full rank. If $x$ minimizes $\|Ax - b\|_2$, then let $r = Ax - b$ be the residual. Let $\tilde{x}$ minimize $\| (A + \delta A) \tilde{x} - (b + \delta b) \|_2$. If $\varepsilon = \max \left( \frac{\| \delta A \|_2}{\| A \|_2}, \frac{\| \delta b \|_2}{\| b \|_2} \right) < \frac{1}{\kappa_2(A)}$, then

$$\frac{\| \tilde{x} - x \|_2}{\| x \|_2} \leq \varepsilon \cdot \left\{ \frac{2 \cdot \kappa_2(A)}{\cos \Theta} + \tan \Theta \cdot \kappa_2^2(A) \right\} + O(\varepsilon^2)$$

where $\sin \Theta = \frac{\| r \|_2}{\| b \|_2}$ or $\Theta$ is the angle between $b$ and $Ax$.

Proof. We know

$$\tilde{x} = \left( (A + \delta A)^T (A + \delta A) \right)^{-1} (A + \delta A)(b + \delta b).$$

Expand this in $\delta A$ and $\delta b$ and consider only linear terms. $\square$

It turns out that there's a nicer form here for the error in the residual.
\[
\frac{\| \tilde{r} - r \|_2}{\| r \|_2} \leq (1 + 2 \epsilon x_2(A)).
\]