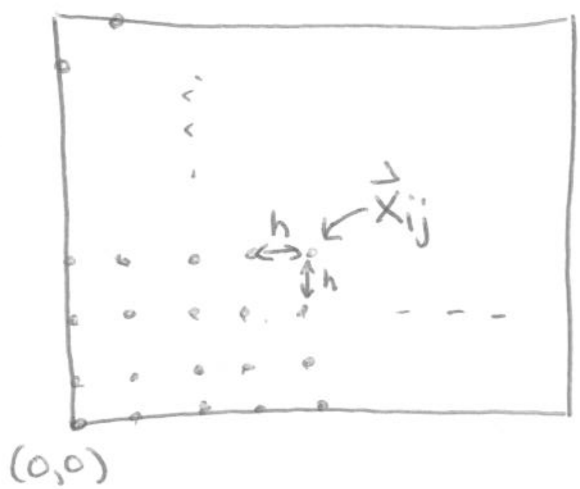
Grid Methods for Laplace / Poisson.

We now introduce our first examples of numerical methods for PDE. Suppose the domain is discretized into a grid with size h.



We let locations on the grid be written

$$\vec{x}_{ij} = (ih, jh)$$

and let our unknown function

$$u(\vec{x}_{ij}) = \{u_{ij}\}$$

one variable per grid pt.

Now consider the problem

$$\Delta u + f u = g$$

with Dirichlet boundary conditions

$$u(x,y) = h(x,y) \quad \text{on } \partial\Omega.$$

We will let

$$f_{ij} = f(\vec{x}_{ij}), \quad g_{ij} = g(\vec{x}_{ij}).$$

Now we can approximate Δu by using our difference formula:

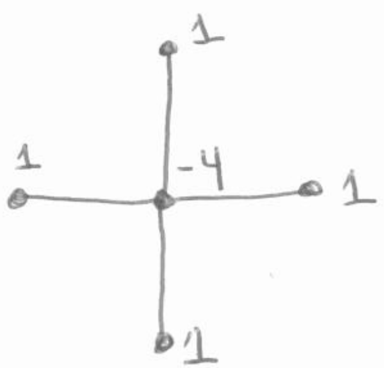
$$\left(\frac{\partial^2}{\partial x^2} u\right)(\vec{x}_{ij}) \approx \frac{1}{h^2}\left(u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right)$$

$$\left(\frac{\partial^2}{\partial y^2} u\right)(\vec{x}_{ij}) \approx \frac{1}{h^2}\left(u_{i,j+1} - 2u_{ij} + u_{i,j-1}\right)$$

so

$$(\Delta u)(\vec{x}_{ij}) \approx \frac{1}{h^2}\left[u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4\,u_{ij}\right]$$

This is called a "five point formula" or "five point stencil" for Δu and is often summarized in diagram form as



or

$$\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}$$

The error in this formula comes from our formula for the second ~~partial~~ derivative error:

$$E = -\frac{h^2}{12}\left[\frac{\partial^4}{\partial x^4}u(\vec{x}_{ij} + (\xi,0)) + \frac{\partial^4}{\partial y^4}u(\vec{x}_{ij} + (0,\xi'))\right]$$

There are certainly other stencils coming from better 2nd derivative estimators.

For instance, the nine-point stencil for Δ is:

$$\begin{matrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{matrix} \quad (\text{multiplied by } 1/6h^2)$$

This is only $O(h^2)$ for a general function, but it is $O(h^6)$ for a harmonic function (that is, a function with $\Delta u = 0$) so this is a great stencil for the Laplace problem

$$\Delta u = 0, \text{ fixed boundary conditions.}$$

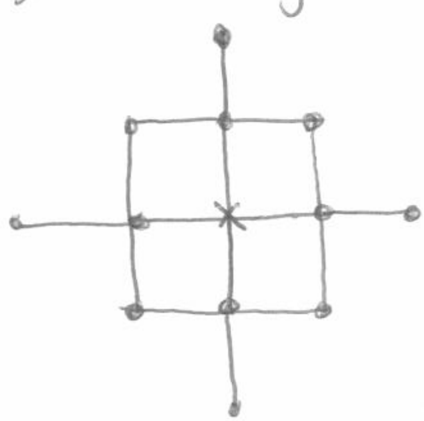
Suppose we use the 5pt stencil for now on the problem

$$\Delta u + fu = g. \quad u = h \text{ on } \partial\Omega$$

We get a system of linear equations in the u_{ij} with f_{ij}, g_{ij}, and h_{ij} entering as constants.

Now you can see a few things about this matrix already

1) u_{ij} is part of 5 stencils



so the column of the matrix for u_{ij} contains

$$\underbrace{4 - h^2 f_{ij}}_{\substack{\text{main stencil} \\ \text{can make this diagonal entry}}} \quad \text{and} \quad \underbrace{4 \ \ -1\text{'s}}_{\substack{\text{other stencils} \\ \text{off-diagonal stuff}}}$$

2) Hence, if $f_{ij} < 0$ then the matrix is diagonally dominant.
Fact: A diagonally dominant matrix is nonsingular!

3) The ~~[illegible]~~ matrix is square.

So we have a very sparse (but not banded) matrix, ~~[illegible]~~ problem which should have a solution.

Solving it will require techniques from our (upcoming) study of numerical linear algebra. To preview, let's suppose we have the equation for u_{ij} (a row):

Assuming we multiplied through by h^2, our equation is

$$-(4-h^2 f_{ij})\,u_{ij} + u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = h^2 g_{ij}$$

Solving for u_{ij}, we get

$$u_{ij} = \frac{1}{4-h^2 f_{ij}}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 g_{ij}\right)$$

Now suppose we use this as an <u>iteration</u>. That is,

$$(u_{ij})^{(k+1)} = \frac{1}{4-h^2 f_{ij}}\left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)} - h^2 g_{ij}\right).$$

If this procedure converges - that is, the vector $(u_{ij}^{(k+1)}) = (u_{ij}^{(k)})$, then this must be the solution to our set of linear equations. (!)

We notice that for

$$\Delta u = 0$$

this method becomes:

replace each u_{ij} with the average of its neighboring values

Clearly if this converges, it converges to a function for which each u_{ij} is the average of its neighbors. But such a function is harmonic.