We want to show that harmonic functions obey a really amazing property:

\[ \Delta u = 0 \text{ on } \Omega \]
with \( u = f \text{ on } \partial \Omega \) \iff \( u \) is the minimizer of the functional

\[ D(v,v) = \int_{\Omega} \nabla v \cdot \nabla v \, d\text{Vol} \]

among functions with \( u = f \text{ on } \partial \Omega \).

This will lead us to some new numerical methods for the Laplace, and in general for elliptic PDE problems.
Green's Identity 1. For functions $u$ and $v$ on a domain $\Omega$ with smooth boundary $\partial \Omega$:

$$\int_{\partial \Omega} (v \nabla u) \cdot \vec{n} \, d\text{Area} = \int_{\Omega} v \Delta u + \nabla u \cdot \nabla v \, d\text{Vol}.$$  

We compute

$$\nabla \cdot (v \nabla u) = \sum_i \frac{\partial}{\partial x_i} (v \nabla u)_i = \sum_i \frac{\partial}{\partial x_i} (v \frac{\partial u}{\partial x_i}) = \sum_i (\frac{\partial v}{\partial x_i}) \cdot (\frac{\partial u}{\partial x_i}) + v \frac{\partial^2 u}{\partial x_i^2} = \nabla v \cdot \nabla u + v \Delta u.$$  

Now this is just the divergence theorem.
Green's Identity 2. If \( \Omega \) is a domain with smooth boundary \( S \), then

\[
\int_S \mathbf{v} (\nabla u \cdot \mathbf{n}) - u (\nabla v \cdot \mathbf{n}) \, d\text{Area} = \\
= \int_\Omega (\mathbf{v} \Delta u - u \Delta v) \, d\text{Vol}
\]

(Just switch \( u \) and \( v \) in identity 1 and subtract.)
Now suppose $u, v$ both equal $f$ on $\Omega$, $\Delta u = 0$. We have

\[
D(u-v,u+v) = \int_\Omega (\nabla u - \nabla v) \cdot (\nabla u + \nabla v) \, d\text{Vol}
\]

\[
= D(u,u) - D(v,v)
\]

But

\[
\int_\Omega \nabla (u-v) \cdot \nabla (u+v) \, d\text{Vol} = \left\langle \text{Green's Id 1.} \right\rangle
\]

\[
= \int_\Omega (u-v) (\nabla (u+v) \cdot \vec{n}) \, d\text{Area} - \int_\Omega (u-v) \Delta (u+v) \, d\text{Vol}
\]

\[
= -\int_\Omega (u-v) \Delta (u-v) \, d\text{Vol} \quad \text{since } \Delta u = 0
\]

\[
= -\int_\Omega \nabla (u-v) \cdot \nabla (u-v) \, d\text{Vol} + \int_\Omega (u-v) \nabla (u-v) \cdot d\vec{A}
\]

\[
\leq 0.
\]
This proves that for any \( v \) on \( \Omega \) obeying the same boundary conditions,

\[ D(u,u) \leq D(v,v) \]

if \( \Delta u = 0 \) on \( \Omega \).

Remarks.

1) This does not prove that such a \( u \) exists, but we could do this properly (e.g., 2F of Folland) and prove that as well.

2) We can use this to obtain specific solutions in many cases.
3) There's a pretty picture here:

\[ D(y, u) \]
is a norm on a space of functions on \( \Omega \)

any function with given boundary data \( f \)

harmonic functions

has a unique orthogonal projection onto harmonic fns.

\( v \) - functions that vanish on \( 2\Omega \)

4) Actually, we could do this for a general elliptic problem (but the details of \( D \) would change, cf chapter 7 of Folland).

5) We need one more case:

\[ \Delta u = r \quad \iff \quad u \text{ minimizes} \]

\[ D(u, u) = \int \Delta u \cdot \Delta u + 2ru \, d\Omega \]
A (better) method of finite elements.

We now use this observation to build a better solver.

Idea: Suppose we triangulate \( \Omega \) and allow values at mesh points to vary, assuming \( u \) is piecewise linear on triangles. We could take

\[
\text{mesh point} \quad (x_i, y_i, z_i)
\]

\[
\rightarrow \text{now try to write down } \Delta u \text{ for vertex in terms of mesh vals.}
\]

(old approach)
(new idea)

If $u$ is linear on this triangle, $\nabla u$ is constant.

Write down

$$\sum \int_{\Omega_i} \nabla u : \nabla u \, dV$$

as a linear function of the vertices of the triangle and minimize the function

$$\sum \int_{\Omega_i} \nabla u : \nabla u \, dV$$

This is (a) finite element method and we will see it's pretty powerful.