Gaussian Elimination II.

We stated last time:

Theorem. If A is nonsingular then there exist permutations P_1 and P_2, a unit lower triangular L and an upper triangular nonsingular U so that $P_1 A P_2 = LU$.

Proof. We proceed by induction on n; the base case is easy, suppose the theorem holds for $(n-1)\times(n-1)$ matrices. Now if A is nonsingular, we can choose P_1, P_2 so that the upper left entry of A is nonzero.

Actually, we only need one of P_1, P_2 to do this, since every row and every column of A ~~have~~ must have a nonzero entry.

Now we write

$$P_1' A P_2' = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ L_{21} & I \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & U_{12} \\ L_{21} u_{11} & L_{21} U_{12} + \tilde{A}_{22} \end{bmatrix}$$

using our usual trick of writing down the decomposition we wish for, then solving for what the pieces have to be.

We get

$$u_{11} = a_{11} \neq 0,$$

$$U_{12} = A_{12}$$

$$L_{21} = \frac{A_{21}}{a_{11}}$$

and so we can solve

$$L_{21} U_{12} + \tilde{A}_{22} = A_{22} \quad \text{as} \quad \tilde{A}_{22} = A_{22} - \frac{A_{21} A_{12}}{a_{11}}$$

Now

$$\det P_1' A P_2' = 1 \cdot u_{11} \cdot \det \tilde{A}_{22} \neq 0,$$

so

$$\det \tilde{A}_{22} \neq 0.$$

Thus $\tilde{A}_{22}$ is a nonsingular $(n-1)\times(n-1)$ block matrix so there exist $\tilde{P}_1$ and $\tilde{P}_2$ so that $\tilde{P}_1 \hat{A}_{22} \tilde{P}_2 = \tilde{L}\tilde{U}$. We now substitute in and compute.

$$P_1' A P_2' = \begin{bmatrix} 1 & 0 \\ L_2 & I \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & \tilde{P}_1^T \tilde{L} \tilde{U} \tilde{P}_2^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ L_2 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_1^T \tilde{L} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & \tilde{U} \tilde{P}_2^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ L_2 & \tilde{P}_1^T \tilde{L} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \tilde{P}_2 \\ 0 & \tilde{U} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_2^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_1^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tilde{P}_1 L_{21} & \tilde{L} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \tilde{P}_2 \\ 0 & \tilde{U} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_2^T \end{bmatrix}$$

Now the left and right matrices are permutation matrices, so we can write things as

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_1 \end{bmatrix} P_1'\right) A \left(P_2' \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}\right) = LU. \quad \square.$$

Corollary. We can choose $P_2' = I$ and P_1' so that a_{11} is the largest entry (in absolute value) in its column.

Then $L_{21} = \dfrac{A_{21}}{a_{11}}$ has entries with absolute value ≤ 1.

Further, when computing L, we can use this trick to make sure that at each step we permute the remaing rows of the matrix to put the largest elt. on the diagonal.

This procedure is called

"Gaussian elimination with partial pivoting"

or

GEPP

Corollary. We can choose P_1' and P_2' so a_{11} is the largest entry (in abs. val.) in the entire matrix.

This alternative procedure is called

"Gaussian elimination with complete pivoting"

or

GECP

We can now write down the proof of our theorem (it <u>was</u> constructive!) as an algorithm.

Algorithm: LU Factorization

for $i = 1$ to $n-1$

(assume we have L and U under construction and we know <u>columns</u> $1 \ldots i\text{-}1$ of L and <u>rows</u> $1 \ldots i\text{-}1$ of U at this point).

GEPP: Permute rows $i..n$ of A by swapping row i with the row containing the largest elt in column i of A. Permute L by the same swap (it stays lower triangular).

GECP: Swap rows of A and L and coloms of A and U to get largest entry in the submatrix of A ~~with~~ ~~row i and~~ in the lower right corner with entries $(i..n) \times (i..n)$.

Now compute column i of L by

$L_{21} = \dfrac{A_{21}}{a_{11}}$ or (at this step) $l_{ji} = a_{ji} / a_{ii}$
for $j = i+1$ to n.

Then compute row i of U by

$$U_{12} = A_{12} \quad \text{or} \quad u_{ij} = a_{ij}, \quad j = i+1 \ldots n.$$

Last, we update the block submatrix A_{22} by

$$\tilde{A}_{22} = A_{22} - \frac{A_{21} A_{12}}{a_{11}}$$

or

for $j = i+1$ to n
  for $k = i+1$ to n
    $a_{jk} = a_{jk} - l_{ji} \cdot u_{ik}$

$\square$

We can count operations (roughly) by noting that the innermost two loops are doing $(n-i)^2$ operations and that this happens for all i in $1 \ldots n$, for a total of $O(n^3)$ operations.

It's worth pointing out that there are some simplifications which can be done (see 2.3 of Demmel for details).