Predictor-Corrector Methods.

Here's another way to think about solving the ODE

$$x'(t) = f(t, x(t)).$$

If want to step from t to $t+h$, observe

$$x(t+h) - x(t) = \int_t^{t+h} x'(t)\,dt$$
$$= \int_t^{t+h} f(t, x(t))\,dt.$$

So if we could approximate this integral, we could derive a rule to compute the next step.

Idea. If we're in the middle of a solution, we know $x(t)$ at

$$t, t-h, t-2h, t-3h, \ldots, t-ph$$

we can use all that data to interpolate a polynomial over

$$f(t, x(t)),\ f(t-h, x(t-h)), \ldots, f(t-ph, x(t-ph))$$

and integrate the resulting polynomial over $(t, t+h)$. This is called an Adams-Bashforth method. For example,

$p=0$ $\quad x(t+h) \approx x(t) + h\, f(t, x(t))$

$p=1$ $\quad x(t+h) \approx x(t) + \frac{h}{2}\left[3 f(t, x(t)) - f(t-h, x(t-h))\right]$

$p=2$ $\quad x(t+h) \approx x(t) + \frac{h}{12}\left[23 f(t, x(t)) - 16 f(t-h, x(t-h)) + 5 f(t-2h, x(t-2h))\right]$

It turns out that we will most often use the $p=3$ formula

$$x(t+h) \approx x(t) + \frac{h}{24}\left[55 f(t, x(t)) - 59 f(t-h, x(t-h)) + 37 f(t-2h, x(t-2h)) - 9 f(t-3h, x(t-3h))\right].$$

This formula is called a <u>predictor</u> because it solves for $x(t+h)$ using only past times. We can improve this guess by interpolating at

$$\cancel{t+h, t, t-h, \ldots, t-ph}$$

$$f(t+h, x(t+h)), f(t, x(t)), \ldots, f(t-ph, x(t-ph))$$

using our predicted value of $x(t+h)$ in the interpolation. We can then integrate the resulting degree $p+1$ polynomial over $(t, t+h)$. This is called an Adams-Moulton method or a <u>corrector</u> method since it improves our estimate of $f(t+h)$. We give

$$p=3 \quad x(t+h) \approx x(t) + \frac{h}{24}\Big[9f(t+h, \widehat{x(t+h)}) + 19f(t, x(t)) - 5f(t-h, x(t-h)) + f(t-2h, x(t-2h))\Big]$$

where $\widehat{x(t+h)}$ is our estimate from predictor.

Together, these two ~~methods~~ formulae form the fourth order Adams-Bashforth-Moulton method.

Truncation error for the ~~AB~~ predictor-corrector:

We recall from a long time ago the error formula for polynomial interpolation

$$f(x)-p(x) = f[x_0, x_1, \dots, x_n, x] \prod_{i=0}^{n} (x-x_i)$$

where we assume that $p(x)$ is a polynomial that interpolates f at nodes $x_0, \dots, x_n$, ~~and~~ that x is not a node, and that $f[\quad]$ is the divided difference.

Now for the predictor formula, we have if we assume that our previous steps were exact calculations of

$$x'(t), x'(t-h), \dots, x'(t-ph)$$

we have that if Predictor_p is the order p estimate,

$$X(t+h) - \text{Predictor}_p(t+h) = \int_t^{t+h} X'[t, t-h, \ldots, t-ph, s] \prod_{i=0}^{p} (s-(t-ih))\, ds$$

Now the function

$$\prod_{i=0}^{p} (s-(t-ih)) \geq 0 \quad \text{for} \quad s \in (t, t+h)$$

so there is some point $\xi \in (t, t+h)$ so that this integral is equal to

$$= X'[t, t-h, \ldots, t-ph, \xi] \int_t^{t+h} \prod_{i=0}^{p} (s-(t-ih))\, ds$$

Now writing $s = t+u$, we see

$$\int_t^{t+h} \prod_{i=0}^{p} (s-(t-ih))\, ds = \int_0^{h} \prod_{i=0}^{p} (u+ih)\, du$$

and writing $u = hv$, we see

$$\int_0^{h} \prod_{i=0}^{p} (u+ih)\, du = \int_0^{1} \prod_{i=0}^{p} (hv+ih) \cdot h\, dv$$

$$= h^{p+2} \int_0^{1} \prod_{i=0}^{p} (v+i)\, dv = \gamma_{p+2}\, h^{p+2}$$

where we may as well define

$$\gamma_{p+2} = \int_0^1 \prod_{i=0}^{p} (v+i)\, dv.$$

Now the divided difference

$$x'[t, t-h, \ldots, t-ph, \xi] = \frac{1}{(p+1)!} (x')^{(p+1)} (\xi_0)$$

$$= \frac{1}{(p+1)!} x^{(p+2)} (\xi_0)$$

for some point ξ_0 in $[t-ph, t+h]$.

So in general if we have a bound on the $(p+2)$nd derivative of a solution curve, we expect truncation error to look like

$$x(t+h) - \text{Predictor}_p (t+h) \approx O(h^{p+2}),$$

which is pretty cool! The truncation error for the corrector is of the same order in h, but the coefficient turns out to be considerably better.

Next time:

RK4 vs Predictor-Corrector showdown.

<u>And</u> the start of systems of ODE.

(surprisingly, it's pretty much the same).