Introduction to PDE

This lecture contains a supercondensed summary of the theory of partial differential equations. This is impossible (one could give a semester course on the basics of PDE), but at the least we will learn some of the basic facts.

Definition. A partial differential equation of order K is an equation in the form

$$F(x, (\partial^\alpha u)_{|\alpha| \leq K}) = 0$$

where $u(\vec{x})$ is a scalar function on $\mathbb{R}^n$ and α is a multi-index representing partial derivatives of order $\leq K$.

There are ~~two~~ a few classes to consider:

1) A PDE is <u>linear</u> if F is linear in u and its partials, or we can write the equation as

$$\sum_{|\alpha| \leq K} a_\alpha(\vec{x})\, \partial^\alpha u = f(\vec{x})$$

where the a_α are some coefficient functions depending on position ($\vec{x}$) but <u>not</u> on the function u.

2) A PDE is <u>quasi-linear</u> if it is linear in the highest order partials of u. That is, we can write the equation as

$$\sum_{|\alpha| = K} a_\alpha(\vec{x}, (\partial^\beta u)_{|\beta|<K})\, \partial^\alpha u = b(\vec{x}, (\partial^\beta u)_{|\beta|<K})$$

3) Nonlinear pde are the remaining equations. Unfortunately, they are common, interesting, and very hard to analyze.

For linear pd equations, the behavior of the coefficients of the highest order partials is a Key to understanding the PDE.

Definition. A linear pde is written

$$\sum_{|\alpha| \leq K} a_\alpha(\vec{x})\, \partial^\alpha u = f(\vec{x}) \iff L u = f$$

where L is said to be a <u>linear differential operator</u> of order K.

Each L has an associated degree K

homogenous polynomial called the characteristic form

$$\chi_L(\vec{x},\vec{v}) = \sum_{|\alpha|=k} a_\alpha(\vec{x})\,\vec{v}^{\alpha}$$

where $\vec{v}^{\alpha} = v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_n^{\alpha_n}$, as usual. The zero set of this polynomial is called the characteristic variety at $\vec{x}$,

$$\mathrm{char}_x(L) = \{\vec{v} \neq \vec{0} \mid \chi_L(\vec{x},\vec{v}) = 0\}.$$

Lemma. If we change coordinates by a map f, $\mathrm{char}_x(L)$ changes by Df.

So suppose $\vec{v} \in \mathrm{char}_x(L)$. By changing coords, we can arrange $\vec{v} = \vec{e}_i$. But

$$\chi_L(\vec{x},\vec{e}_i) = 0 \iff a_\alpha(\vec{x}) = 0 \text{ for}$$

$$\alpha = \underbrace{(0,\dots,k,\dots 0)}_{i\text{th position}} \text{ "}= k\vec{e}_i\text{"}.$$

That is, the operator L does <u>not</u> contain the term $\frac{\partial^k}{\partial x_i^k} u$. In a sense,

$$\boxed{\text{a linear operator of order } k \text{ is ``not really order } k \text{ in direction } \vec{v}\text{'' if } \vec{v} \in \mathrm{char}_x(L).}$$

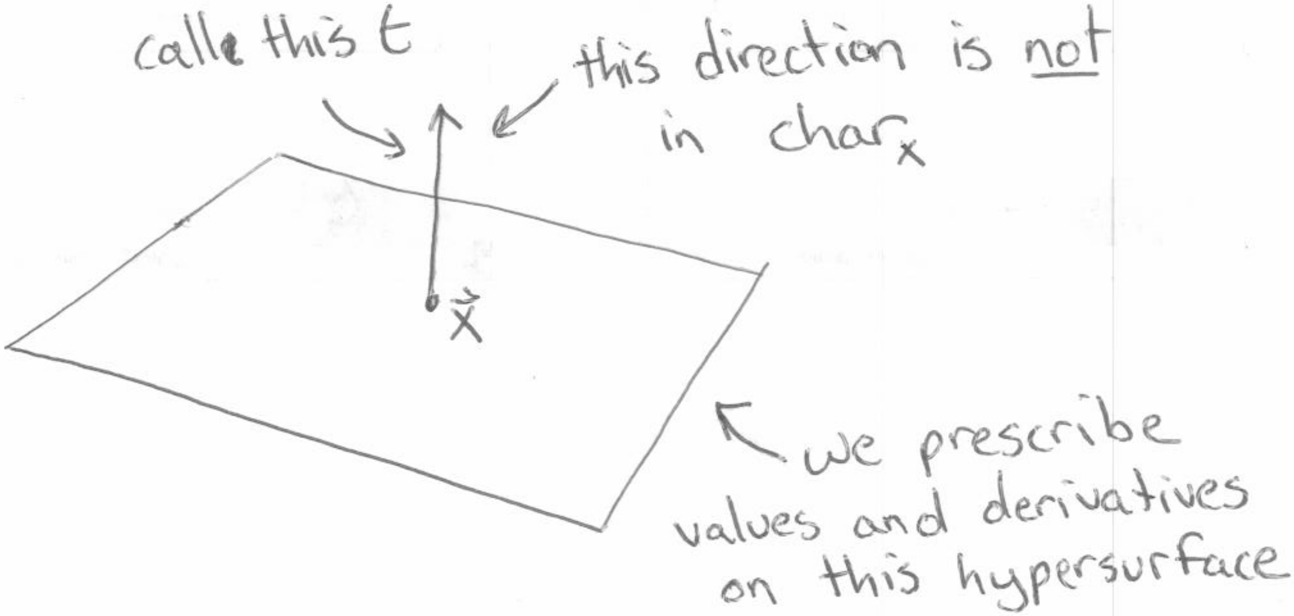
Definition. A linear operator is <u>elliptic</u> at $\vec{x}$ if $\mathrm{char}_x(L) = \emptyset$.

Generally speaking,

$$L \text{ elliptic} \Longrightarrow \text{you happy.}$$

(the reverse implication sadly does not hold).

Is there an existence theorem for PDE? Not a very generally useful one. We have something like



Cauchy-Kovaleskaia Theorem:* If ~~the PDE is linear enough that~~ we can solve $F=0$ for $\partial^k/\partial t^k \, u$ and everything is analytic then $\exists$ a unique analytic solution in a neighborhood of $\vec{x}$.

For linear pde, we have the Holmgren uniqueness thm, which says there are no additional non-analytic solutions.

Example operator.

$\Delta = \sum_j \partial_j^2$ is a linear, elliptic operator.

This is called the Laplacian.

Theorem. Suppose L is a partial differential operator which commutes with translations and rotations in $\mathbb{R}^n$. Then L is a polynomial in Δ.

Next time. Example equations.
Parabolic and hyperbolic equations.
Boundary data.