Introduction to PDE

This lecture contains a supercondensed summary of the theory of partial differential equations. This is impossible (one could give a semester course on the basics of PDE), but at the least we will learn some of the basic facts.

Definition. A partial differential equation of order $K$ is an equation in the form

$$F(x, (D^k u)_{1 \leq k \leq K}) = 0$$

where $u(x)$ is a scalar function on $\mathbb{R}^n$ and $\alpha$ is a multi-index representing partial derivatives of order $\leq K$. 
There are a few classes to consider.

1) A PDE is linear if $F$ is linear in $u$ and its partials, or we can write the equation as

$$\sum_{|\alpha| \leq K} a_\alpha(x) \partial^\alpha u = f(x)$$

where the $a_\alpha$ are some coefficient functions depending on position $(x)$ but not on the function $u$.

2) A PDE is quasi-linear if it is linear in the highest order partials of $u$. That is, we can write the equation as

$$\sum_{|\alpha| = K} a_\alpha(x, (\partial^B u)_{1 \leq B \leq K}) \partial^\alpha u = b(x, (\partial^B u)_{1 \leq B \leq K})$$
3) Nonlinear pde are the remaining equations. Unfortunately, they are common, interesting, and very hard to analyze.

For linear equations, the behavior of the coefficients of the highest order partials is a key to understanding the PDE.

Definition. A linear pde is written

$$\sum_{|\alpha| \leq K} a_\alpha(x) \partial^\alpha u = f(x) \iff Lu = f$$

where $L$ is said to be a linear differential operator of order $K$.

Each $L$ has an associated degree $K$. 
homogeneous polynomial called the characteristic form

\[ X_L(\vec{x}, \vec{v}) = \sum_{|\alpha| = k} a_\alpha(\vec{x}) \vec{v}^\alpha \]

where \( \vec{v}^\alpha = v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_n^{\alpha_n} \) as usual.

The zero set of this polynomial is called the characteristic variety at \( \vec{x} \),

\[ \text{char}_x(L) = \{ \vec{v} \neq 0 | X_L(\vec{x}, \vec{v}) = 0 \} \]

Lemma. If we change coordinates by a map \( f \), \( \text{char}_x(L) \) changes by \( Df \).

So suppose \( \vec{v} \in \text{char}_x(L) \). By changing coords, we can arrange \( \vec{v} = \vec{e}_i \). But

\[ X_L(\vec{x}, \vec{e}_i) = 0 \iff a_\alpha(\vec{x}) = 0 \text{ for } \alpha = (0, \ldots, k, \ldots, 0) \text{ in the } i\text{th position} \]
That is, the operator $L$ does not contain the term $\frac{\partial^k u}{\partial x_i^k}$. In a sense, a linear operator of order $K$ is "not really order $K$ in direction $\nabla$" if $\nabla \in \text{char}_x(L)$.

Definition. A linear operator is elliptic at $\bar{x}$ if $\text{char}_x(L) = \emptyset$.

Generally speaking,

$L$ elliptic $\iff$ you happy.

(the reverse implication sadly does not hold.)
Is there an existence theorem for PDE? Not a very generally useful one. We have something like

call this $\mathcal{E}$

dhis direction is not in char $x$

we prescribe values and derivatives on this hypersurface

**Cauchy-Kovaleskaya Theorem.** If the PDE is linear enough that we can solve $F=0$ for $\frac{\partial^k u}{\partial x^k}$ and everything is analytic then there is a unique analytic solution in a neighborhood of $\tilde{x}$.

For linear PDE, we have the Holmgren uniqueness thm, which says there are no additional non-analytic solutions.
Example operator.

\[ \Delta = \sum_j \partial_j^2 \] is a linear, elliptic operator.

This is called the Laplacian.

Theorem. Suppose \( L \) is a partial differential operator which commutes with translations and rotations in \( \mathbb{R}^n \). Then \( L \) is a polynomial in \( \Delta \).

Next time. Example equations.
Parabolic and hyperbolic equations.
Boundary data.