Chapter 9 - Independence and Conditioning

Definition. Given a pair of discrete random variables $X, Y$, the joint pmf is

$$P_{X,Y}(x,y) = P(X=x \text{ and } Y=y)$$

The joint CDF is

$$F_{X,Y}(x,y) = P(X \leq x \text{ and } Y \leq y)$$

Example. Roll 2 dice, let $X$ be the minimum and $Y$ be the maximum of the two values.

$$P_{X,Y}(3,5) =$$

$$P_{X,Y}(5,3) =$$

$$P_{X,Y}(4,4) =$$
Example. Flip a coin 3 times and let $X =$ # of heads, $Y =$ # of tails.

\[ P_{X,Y}(3,0) = \frac{1}{8} = \{H,H,H,HT\} \]
\[ P_{X,Y}(2,1) = \frac{3}{8} = \{T,H,H,HT,T,H,T,HT\} \]
\[ P_{X,Y}(1,2) = \frac{3}{8} \]
\[ P_{X,Y}(0,3) = \frac{1}{8} \]

If we are given a joint distribution of two variables, $P_{X,Y}$, we may extract the marginal distributions of one by

\[ P_X(x) = \sum_y P_{X,Y}(x,y) \]
\[ P_Y(y) = \sum_x P_{X,Y}(x,y) \]

Example. If we roll two dice and $X = \min$, $Y = \max$

\[ P_X(1) = \sum_{y=1}^{6} P_{X,Y}(1,y) \]
\[ = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{2}{36} + \frac{2}{36} + \frac{2}{36} \]
\[ P_X(2) = \sum_{y=1}^{6} P_{X,Y}(2,y) = 0 + \frac{1}{36} + \frac{2}{36} + \frac{2}{36} + \frac{2}{36} + \frac{2}{36} = \frac{9}{36} \]

\[ P_X(3) = 0 + 0 + \frac{1}{36} + \frac{2}{36} + \frac{2}{36} + \frac{2}{36} = \frac{7}{36} \]

\[ P_X(4) = 0 + 0 + 0 + \frac{1}{36} + \frac{2}{36} + \frac{2}{36} = \frac{5}{36} \]

\[ P_X(5) = \frac{3}{36} \]

\[ P_X(6) = \frac{4}{36} \]

Note that 11 + 9 + 7 + 5 + 3 + 1 = 36, so all the probability mass is accounted for.
Definition. We say that $X, Y$ are independent random variables iff $P_{X,Y}(x,y) = P_X(x)P_Y(y)$.

Example. As above, roll 2 dice and let $X = \min, Y = \max$.

We know $P_X(1) = \frac{1}{36}$. It is not hard to see $P_Y(1) = \frac{1}{36}$.

$$P_{X,Y}(1,1) = \frac{1}{36}$$

$$P_X(1)P_Y(1) = \frac{1}{36} \cdot \frac{1}{36}$$

so these r.v.s are not independent.

Example. Roll 1 die and let $X = 1$ if the result is 1, 3, 5 (odd) and $X = 0$ if the result is even. Let $Y = 1$ if the result is 5 or 6 and 0 otherwise.

Compute $P_{X,Y}$ for $(0,0), (0,1), (1,0)$ and $(1,1)$. Also compute $P_X$ for 0, 1 and $P_Y$ for 0, 1.
This is an example of a more general phenomenon.

Definition. $X$ is an indicator for event $A$ if $X = 1$ when $A$ occurs and $X = 0$ when $A^c$ occurs.

Theorem. Let $X, Y$ be indicators for $A, B$. $X, Y$ are independent r.v.s $\iff A, B$ are independent events.

Example. Flip a coin until you get heads. Let $X = 1$ if an even number of flips are needed and $X = 0$ otherwise. Let $Y = 1$ if 11 or more flips are needed and $Y = 0$ otherwise.

Decide if independent.