Definition. Events A and B are independent if \( P(A \cap B) = P(A)P(B) \), and dependent otherwise.

Example. \( \begin{array}{c}
\text{A} \\
\text{B}
\end{array} \quad \begin{array}{c}
P(A \cap B) = \frac{1}{2} P(A) = P(A)P(B)
\end{array} \)

This is easier to draw if \( S \) has a special form. Given any probability space \( S \) with \( n \) equally likely outcomes, we can define the probability of a collection of random events are observed, we call each one a trial, or a draw.

If we take 2 trials from the same space \( S \), the pair \((x_1, x_2)\) is selected from \( S \times S \).
In this case, any event in the form $\mathcal{E}(x_1, x_2) \in \mathcal{S} \times \mathcal{S} | x_1 \in A_3$ and any event in the form $\mathcal{E}(x_3, x_2) \in \mathcal{S} \times \mathcal{S} | x_2 \in B_3$ are independent.

Example. We flip a (fair) coin twice. Here are three events:

A = $\mathcal{E}(x_1, x_2) | x_1 = H_3$
B = $\mathcal{E}(x_1, x_2) | x_2 = H_3$
C = $\mathcal{E}(x_1, x_2) | x_1 = x_2 = H_3$
D = $\mathcal{E}(x_1, x_2) | x_1 = x_2 = T_3$
Which pairs of $A, B, C, D$ are independent?

Example. We roll a fair (6-sided) die.

$A = \{1, 2, 3\}$

$B = \{3, 4\}$

$C = \{1, 2, 3, 5, 6\}$

Which pairs are independent?

Lemma. $A$ and $B$ are independent and disjoint $\iff P(A) \text{ or } P(B) = 0$ (or both)

Theorem. If $A \cap B$ and $P(A) \neq 0$ and $P(B) \neq 1$, then $A, B$ are dependent.

Proof. $P(A \cap B) = P(A)$ so $P(A \cap B) P(B) = P(A) P(B)$
Theorem. If neither $P(A) = 0$ nor $P(A) = 1$ then $A, A^c$ are dependent.

We can extend the definition of independence to multiple events.

Definition. A collection of events $A_j$ is (mutually) independent if for every subcollection $C \subseteq$ of the $A_j$,

$$P(\bigcap A_j) = \prod_{j \in C} P(A_j)$$

Note. This is true for any collection of trials as long as $A_j$ (i.e., $\{x_1, \ldots, x_j, \ldots\}$) as long as event $A_j$ only depends on $x_j$. 
Example. We flip a coin repeatedly. The individual flips are independent. (Note. The coin doesn't have to be fair)

Example. Kroger sells bags of onions which contain 1 bad onion in each bag.

A = an onion picked from bag A is bad
B = an onion picked from bag B is bad
C = a second onion picked from bag A is bad.

Which of these events are independent?

Example. We flip a coin until a total of 10 heads appear. $A_k = \# \text{flips between } k\text{th head and } (k+1)\text{st head} > 23$.

Are the $A_k$ independent? (Discuss).
Example. A student rolls 7 (fair) dice, yielding numbers \( (x_1, \ldots, x_7) \), each between 1 and 6.

\[
A = x_7 \text{ is even (}\#\text{ on die 7 is even)}
\]

\[
B = x_{(x_7)} \text{ is even (}\# \text{ on die (whatever}\ # \\
\text{ was rolled on die 7) is even)}
\]

Are \( A \) and \( B \) independent? (Discuss.)

Example. A student rolls 3 dice which are unfair \( (P(1) = \frac{1}{3}, P(2) = 0, P(3) = P(4) = P(5) = P(6) = \frac{1}{6}) \) and 4 dice which are fair, producing 7 numbers \( (x_1, \ldots, x_7) \). The \( 3 \) unfair dice are ordered (unfair, unfair, unfair, fair, fair, fair, fair, fair).
Are the two events above dependent or independent? (Discuss).