Interlude: The Ruin Problem.

We will now take a brief break to do something harder, using the ideas we’ve developed so far.

Q: Suppose two players engage in a betting game where player I starts with x dollars and player II starts with N-x dollars.

At each round, they bet 1 dollar, and player I wins with probability p. The game ends when one player or the other runs out of money.

What is the probability $P(x)$?

$P(x) = P(\text{player I wins} | x)$?
We are going to see this problem as a graph

\[ 0 \rightarrow 1 \rightarrow K \rightarrow N-1 \rightarrow N \]

where node \( K \) represents the state "player I has \( K \) dollars."

We define

\[ P_k = \text{probability of moving to state node } k+1 \text{ from node } K. \]

Suppose there exist numbers \( C_k \) so that

\[ P_k = \frac{C_{k+1}}{C_k + C_{k+1}} \]

and let \( r_k = \frac{1}{C_k} \).
Example. If all $p_k = \frac{1}{2}$, all $C_k = 1$ is an ok solution, as

$$P_k = \frac{C_{k+1}}{C_k + C_{k+1}} = \frac{1}{1+1} = \frac{1}{2}$$

For other $p_k$, we'll show later how to we observe that if $q_k = 1 - p_k$, then

$$q_k = 1 - \frac{C_{k+1}}{C_k + C_{k+1}}$$

$$= \frac{C_k + C_k - C_{k+1}}{C_k + C_{k+1}}$$

$$= \frac{C_k}{C_k + C_{k+1}}$$

So

$$\frac{p_k}{q_k} = \frac{C_{k+1}}{C_k}$$
This means that

\[ C_2 = C_1 \frac{p_1}{q_1} \]

\[ C_3 = C_1 \frac{p_1 p_2}{q_1 q_2} \]

\[ C_{n-1} = C_1 \frac{p_1 \cdots p_{n-2}}{q_1 \cdots q_{n-2}} \]

Now we're going to do something cool! Suppose our graph is composed of resistors with resistance \( R_k \) and conductance \( C_k \), and we add a battery so that the voltage \( v(x) \) has

\[ v(0) = 0, \quad v(N) = 1. \]

We will show that \( p(x) = v(x) \) everywhere, and use this to compute \( p(x) \) explicitly!
Step 1. \( p(0) = 0, \ p(N) = 1. \)

This is obvious. At 0, player I has no money left to bet, so they cannot win. At \( N \), player I has all the money, so they cannot lose.

To prove this in the middle, we need some new ideas.

Definition. Given \( C_k \) associated to the edges of a linear graph, we say \( f(x) \) is harmonic if

\[
f(x) = \frac{C_{x+1}}{C_x + C_{x+1}} \ f(x+1) + \frac{C_x}{C_x + C_{x+1}} \ f(x-1)
\]

That is, \( f(x) \) is a weighted average of \( f(x+1) \) and \( f(x-1) \).
Step 2. \( p(x) \) is harmonic.

We know that if player I has \( x \) dollars at some point, in the next round, \( p_I \) will have either \( x+1 \) or \( x-1 \) dollars.

\[
p(x) = P(p_I \text{ wins } | p_I \text{ has } x \text{ dollars})
= p_k p(x+1) + q_k p(x-1)
= P(p_I \text{ wins next game}) \cdot P(p_I \text{ wins}) \cdot P(p_I \text{ wins next game})
+ P(p_I \text{ loses next game}) \cdot P(p_I \text{ wins } | p_I \text{ has } x \text{ dollars and loses next game})
= \frac{C_{k+1}}{C_k + C_{k+1}} p(x+1) + \frac{C_k}{C_k + C_{k+1}} p(x-1).
\]
Step 3. $v(x)$ is harmonic.

Kirchhoff's current law says that

\[ i_x = i \]

the current $i$ flowing into and out of node $x$ is the same.

Ohm's law says

\[ i = \frac{v(x) - v(x-1)}{r_x} \]

Recalling that $r_x = \frac{1}{C_x}$,
\[
C_x v(x) - E_x v(x-1) = C_{x+1} v(x+1) - C_{x+1} v(x)
\]

\[
(C_x + C_{x+1}) v(x) = C_{x+1} v(x+1) + C_x v(x-1)
\]

\[
v(x) = \frac{C_{x+1} v(x+1) + C_x v(x-1)}{C_x + C_{x+1}}.
\]

Step 4. The maximum and minimum of a harmonic function are attained at the boundary.

Proof. Suppose \( f(x) = M \). Then we claim \( f(x-1) = M = f(x+1) \), for otherwise

\[
M = f(x) = \frac{C_{x+1} f(x+1) + C_x f(x-1)}{C_x + C_{x+1}}
\]

\[
\leq \frac{C_{x+1} M + C_x M}{C_x + C_{x+1}} = M.
\]
Step 5. If \( f, g \) are harmonic, then \( f + g \) and \( k f \) are harmonic.

Proof. Just regroup the definitions.

Step 6. If \( f, g \) are harmonic and \( f(0) = g(0), f(N) = g(N) \) then \( f(x) = g(x) \).

By Step 5, \( f - g = h \) is a harmonic function with \( h(0) = h(N) = 0 \).
By step 4, this means the max and min of \( h \) are 0, so \( h(x) = 0 \) everywhere.

We now know \( \rho(x) = u(x) \).
We can use this to finish the problem.

\[
\begin{align*}
\rho_{ox} &= \frac{\xi}{x} \\
\rho_{ox} &= \frac{\xi}{N - x}
\end{align*}
\]
We know

\[
\frac{V(x) - V(0)}{r_{0x}} = i = \frac{V(N) - V(x)}{r_{nx}}
\]

\[\Box\]

and \(r_{0x} = r_1 + \ldots + r_{x+1}\) while \(r_{nx} = r_{x+1} + \ldots + r_{N-1}\). So, using \(V(0) = 0, V(N) = 1\), we get

\[
V(x) = 1 - \frac{1}{\sum_{i=1}^{x+1} r_i}
\]

or

\[
V(x) = \frac{\sum_{i=1}^{x+1} r_i}{\sum_{i=1}^N r_i}
\]

and

\[
\left(\sum_{i=x+1}^{N-1} r_i\right) V(x) = \sum_{i=1}^{x+1} r_i - \left(\sum_{i=1}^{x+1} r_i\right) V(x)
\]
Now if $p = 1/2$, $r_i = C_i = 1$, so we get

$$v(x) = p(x) = \frac{x}{N}.$$ 

In general, $r_i = \frac{1}{C_i}$, and so

$$E_k = \sum r_i = \frac{1}{C_k} \overline{q_1 \cdots q_{i-1}}$$

so

$$p(x) = \frac{1}{C_1} + \frac{1}{C_1} \overline{q_1 \cdots q_{i-1}}$$

$$= \frac{1}{C_1} + \frac{1}{C_1} \frac{p_1}{q_1} + \cdots + \frac{1}{C_1} \frac{p_{i-1}}{q_{i-1}}$$

Exercise: Simplify this formula when $p_1 = \cdots = p_n = p$, $q_1 = \cdots = q_n = q = 1 - p$. 