The Carpenter’s Ruler Theorem.

Suppose we fix edgelengths (thus restricting to a slice of $G_s(\mathbb{R}^n)$ of codimension $(n-1)$) and then further remove configurations where edges cross each other (removing large, top-dimensional chunks). Is what’s left connected? Contractible?

Q. Can every embedded polygon with fixed edgelengths be reconfigured through embedded polygons to a convex polygon?

We will need a few tools.

Definition. A tensegrity is an embedded planar graph $G$ with each edge marked strut, cable, or bar.
The configuration space of a tensegrity consists of configurations embeddings of the same graph, $G'$ where

$$\text{length } e'_i \geq \text{length } e_i, \text{ if } e_i \text{ is a strut}$$

$$\text{length } e'_i = \text{length } e_i, \text{ if } e_i \text{ is a bar}$$

$$\text{length } e'_i \leq \text{length } e_i, \text{ if } e_i \text{ is a cable.}$$

Definition. A tensegrity $\gamma$ is rigid if the configuration space $\nu \gamma \subseteq \nu G$ is a connected component of the copy of $E(2)$ containing $\gamma$.

**Infinitesimal rigidity** is a stronger condition that's easier to detect. An infinitesimal motion of a tensegrity is an assignment of “velocity” vectors $v_i$ to the vertices $p_i$ of $\gamma$. 
so that
\[
\langle (v_i - v_j), p_i - p_j \rangle \geq 0, \text{ if } e_{ij} \text{ is a strut} \\
\langle v_i - v_j, p_i - p_j \rangle = 0, \text{ if } e_{ij} \text{ is a bar} \\
\langle v_i - v_j, p_i - p_j \rangle \leq 0, \text{ if } e_{ij} \text{ is a cable.}
\]

Theorem. If there is a path of configuration of \( G \) starting at \( G(0) = G \), then

\[
v_i = \frac{d}{dt} p_i(t) \bigg|_{t=0}
\]

is an infinitesimal motion. Thus, every infinitesimal motion is a tangent vector to a motion in \( E(2) \)

"infinitesimal rigidity \( \Rightarrow \) rigidity"
The converse is not true, alas. This tensegrity is infinitesimal rigid, but it is rigid. The vectors shown are an i.m.

We can formalize things a little more with the rigidity matrix

\[
\begin{bmatrix}
-\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j - \hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i \\
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{v}}_1 \\
\vdots \\
\dot{\mathbf{v}}_n \\
\end{bmatrix} =
\begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_k \\
\end{bmatrix}
\]

# rows = \# edges
# columns = 2 \cdot \# vertices

Now we see:

infinitesimal rigidity \iff rigidity matrix has rank 2|V|-3
Here's a big idea: suppose we have a linear programming problem $A x \geq 0$. Then

This is called the Farkas alternative theorem: in any either in

$$\text{Im} A \cap (\mathbb{R}^n)^+ = \emptyset \quad \text{or} \quad (\text{Im} A)^\perp \cap (\mathbb{R}^n)^+ = \emptyset$$

$$\text{Im} A \cap (\mathbb{R}^n)^+ \neq \emptyset \implies (\text{Im} A)^\perp \cap (\mathbb{R}^n)^+ = \emptyset$$

(and of course vice versa).
In our case, we like to use the version that Wikipedia calls Gordan's Theorem.

Either $Ax < 0$ has a solution or $A^T y = 0$ has a nonzero solution $y \geq 0$.

For tensegrities, the transpose of the rigidity matrix is the matrix

$$
\begin{bmatrix}
1 & p_i - p_j \\
p_j - p_i & 1
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_k
\end{bmatrix} =
\begin{bmatrix}
\omega_1 & = & \sum (p_i - p_j) \lambda_i \\
\vdots & & \\
\omega_n & = & \sum (p_n - p_j) \lambda_i
\end{bmatrix}
$$

"Stress" on each strut, bar, cable
"Net force" at each vertex.

This is the heart of the stress rigidity theorem.
There is an equ

Definition. We say a set of \( \lambda_{ij} \) is a stress if

\[
\lambda_{ij} \geq 0, \text{ if edge } e_{ij} \text{ is a strut} \\
\lambda_{ij} \leq 0, \text{ if edge } e_{ij} \text{ is a cable}
\]

(\( \lambda_{ij} \) has either sign if \( e_{ij} \) is a bar)

and an equilibrium stress if

\[
\sum_{j} \lambda_{ij}(\dot{p}_i - \dot{p}_j) = 0 \quad \text{for all } i \in \{1, \ldots, n\}.
\]

Theorem. (Roth & Whitely, 1981)
A tensegrity is infinitesimally rigid \( \iff \)

1. it has an equilibrium stress
   which is nonzero on each strut \& cable
2. the corresponding "all bars" linkage is infinitesimally rigid.
Example.

Now we get to something really cool—a connection between surface theory and tensegrity theory.

Definition. A polyhedral lifting of a tensegrity which is a planar graph is an assignment of $z$-heights so that the vertices
around each face lie in a plane, and the exterior face lies in the $z=0$ plane.

Given a lifting, at each edge there are two face normals. Assigning an orientation to the edge, there is a right normal $\hat{n}_r$ and a left normal $\hat{n}_e$.

We say the edge is a mountain, if $\langle \hat{n}_r \times \hat{n}_e, \hat{e} \rangle > 0$
valley, if $\langle \hat{n}_r \times \hat{n}_e, \hat{e} \rangle < 0$.
flat, if $\langle \hat{n}_r \times \hat{n}_e, \hat{e} \rangle = 0$

Note that $\hat{n}_r \times \hat{n}_e$ and $\hat{e}$ are colinear by construction, so the last condition could also be $\hat{n}_e = \hat{n}_r$. 
Maxwell-Cremona Theorem. (1890, 1864, 1982)
A planar tensegrity has an equilibrium stress $\iff$ it has a polyhedral lifting.

Further, a lifting corresponds to a stress which is

positive $\iff$ the edge is a mountain
negative $\implies$ the edge is a valley
zero $\iff$ the edge is flat.