Lecture 3. Operations on vector bundles

Given a vector bundle ξ, there are some canonical constructions that produce new bundles.

Example. If $\overline{B} \subset B$, we can restrict ξ to $\overline{B}$, giving a new bundle $\overline{E}$ with the same fiber, and base $\overline{B}$, and total space $\overline{E} = \pi^{-1}(\overline{B})$.

Example. Given ξ and a map $f: B_1 \to B$, we define the induced bundle $f^*\xi$ over B_1.

$$\begin{array}{ccc} & & E \\ & & \downarrow \pi \\ B_1 & \xrightarrow{f} & B \end{array}$$

For $f^*\xi$ we have:

Base space	: B_1
Total space	: $E_1 \subset B_1 \times E$, (b,e) s.t. $f(b) = \pi(e)$.
Projection	: $\pi_1(b,e) = b$.

We can define

$\hat{f}: E_1 \to E$ by $\hat{f}(b,e) = e$.

This makes

$$\begin{tikzcd} E_1 \arrow[r, "\hat{f}"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\ B_1 \arrow[r, "f"] & B \end{tikzcd}$$

a commutative diagram. Now we note that each

$$\pi_1^{-1}(b) = \pi^{-1}(f(b))$$

= a fiber of the original ξ, which has a vector space structure from the original bundle

We need to show $f^*\xi$ is locally trivial.

a) For any $b \in B_1$, choose a local trivialization (h, U) around $f(b) \in B$. Now $f^{-1}(U) = U_1$ is an open neighborhood in B_1 of b, and we map

$$h_1 : U_1 \times \mathbb{R}^n \to \pi_1^{-1}(U_1) \subset E_1$$

by

$$h_1(b,x) = (b, h(f(b), x)).$$

We must check h_1 a homeomorphism.

1) h_1 is 1-1.

If $(b,x) \neq (b',x')$ then $b \neq b'$ or $x \neq x'$.
In first case, $h_1(b,x) = (b, \sim) \neq h_2(b',x') = (b', \sim)$.
In second, $h_1(b,x) = (b, h(f(b),x))$ and $h_1(b',x') = (b', h(f(b'),x'))$. Now h is 1-1, and $(f(b),x) \neq (f(b'),x')$ since $x \neq x'$, so $h(f(b),x) \neq h(f(b'),x')$.

2) h_1 is onto.

If $(b,e) \in \pi_1^{-1}(U_1)$, then $\pi_1(b,e) \in U_1$ or $b \in U_1$, or $f(b) \in U$. ~~Now~~ ~~I~~ Further, $\pi(e) = f(b)$, so $\pi(e) \in U$. Now h is a homeomorphism from $U \times \mathbb{R}^n \to \pi^{-1}(U)$, ~~so~~ ~~$\exists$~~ with $h(f(b), \sim)$ an isomorphism from $\mathbb{R}^n$ to $\pi^{-1}(f(b))$. Thus $\exists$ $(f(b), x)$ in $U \times \mathbb{R}^n$ so that

$$h(f(b),x) = e.$$

But then $h_1(b,x) = (b, h(f(b),x))$
$= (b,e)$, as desired.

3) h_1 is continuous.

h_1 is a map from $(B_1 \times \mathbb{R}^n)$ to $B_1 \times E$. In the first factor it is the identity, in the second, the composition of f and h. Since f, h are cts, it is cts in both factors.

4) h_1^{-1} is continuous.

As above, $h_1 = Id_{B_1} \times (h \circ f)$. Suppose we have $(b_n, e_n) \to (b, e)$ in $\pi_1^{-1}(U_1)$. Then we observe $b_n \to b$, and

$$e_n = h(\bar{b}_n, x_n), \quad e = h(\bar{b}, x)$$

with $\bar{b}_n \to \bar{b}$ and $x_n \to x$ by continuity of h^{-1}.

Now

$$h_1^{-1}(b_n, h(\bar{b}_n, x_n)) = (b_n, x_n)$$

so this limits to $(b, x) = h_1^{-1}(b, e)$ by construction of x.

Conclusion. This takes untangling, but is basically easy. Skip in lecture.

Remark. If ξ is smooth, and f is too, $E_1 \subset B_1 \times E$ is a smooth submanifold, and $f^*\xi$ is a smooth ~~base~~ vector bundle.

Now we can extend this idea to ~~map~~ define a kind of compatibility between maps bundle $\to$ bundle and maps base $\to$ base.

<u>Definition</u>. A <u>bundle map</u> from η to ξ is a continuous function

$$g : E(\eta) \to E(\xi)$$

which carries each fiber $F_b(\eta)$ isomorphically onto some fiber $F_{b'}(\xi)$. If we define

$$\bar{g}(b) = b',$$

then $\bar{g} : B(\eta) \to B(\xi)$ is continuous.

<u>Lemma</u>. If $g: E(\eta) \to E(\xi)$ is a bundle map and $\bar{g}: B(\eta) \to B(\xi)$ the corresponding map of base spaces, $\eta \cong \bar{g}^*\{\xi\}$.

<u>Proof</u>. Define a map

$$h: E(\eta) \to E(\bar{g}^*\xi)$$

by $h(e) = (\pi(e), g(e))$. h is clearly continuous. Further, on any fiber $F_b(\eta) = \pi^{-1}(b)$, we have

$$\begin{aligned} h(F_b(\eta)) &= (b, g(F_b(\eta))) \\ &= (b, F_{\bar{g}(b)}(\xi)) \end{aligned}$$

and the map on fibers given by taking

$$\begin{tikzcd} F_b(\eta) \arrow[r, "h"] \arrow[rr, bend right, "g"'] & b \times F_{\bar{g}(b)}(\xi) \arrow[r] & F_{\bar{g}(b)}(\xi) \end{tikzcd}$$

is $g|_{F_b(\eta)}$ which is an isomorphism.

We now have a continuous map h between bundles which is an isomorphism on each fiber — by previous Lemma, h is a bundle isomorphism. $\square$

Here's another construction, which is basically trivial.

<u>Definition</u>. Given ξ_1, ξ_2 with projection maps $\pi_i : E_i \to B_i$ the (Cartesian) product bundle is the bundle $\xi_1 \times \xi_2$ with

Total space	$E_1 \times E_2$
Base space	$B_1 \times B_1$
Projection	$\pi_1 \times \pi_2$
Fiber over (b_1, b_2)	$F_{(b_1)}(\xi_1) \times F_{b_2}(\xi_2)$.

Example.

$$T(M_1 \times M_2) = TM_1 \times TM_2$$

Suppose you have two bundles over the same space B. You can combine them in the following way.

Definition. Given ξ_1, ξ_2, vector bundles over B, let $d: B \to B \times B$ be the diagonal embedding and let

$$d^*(\xi_1 \times \xi_2) := \xi_1 \oplus \xi_2$$

be ~~[illegible]~~ called the Whitney sum of ξ_1, ξ_2.

This is, as expected, the bundle with each fiber $F_b(\xi_1 \oplus \xi_2) \cong F_b(\xi_1) \oplus F_b(\xi_2)$.

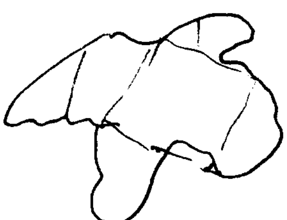
Definition. Given bundles ξ, η over B with $E(\xi) \subset E(\eta)$, we say ξ is a sub-bundle of η if each fiber $F_b(\xi)$ is a subspace of $F_b(\eta)$.

<u>Lemma</u>. If ξ_1, ξ_2 are sub-bundles of η so that each fiber $F_b(\eta)$ is the direct sum of $F_b(\xi_1)$ and $F_b(\xi_2)$. Then $\eta = \xi_1 \oplus \xi_2$.

<u>Proof.</u> Easy. We have a bundle map

$$f : E(\xi_1 \oplus \xi_2) \to E(\eta)$$

given by $f(b, e_1, e_2) = e_1 + e_2$. This is a ~~cts and a~~ bundle isomorphism since $\xi_1 \oplus \xi_2, \eta$ have the same base by Lemma 2.3.



If we know that splitting each fiber into a direct sum splits the entire bundle, we might attempt to come up with a sort of bundle divison (bundle subtraction?).

<u>Definition</u> If $\xi \subset \eta$ are Euclidean vector bundles, then $\xi^\perp$ is the sub-bundle of η whose total space is the union of $F_b(\xi)^\perp$ inside $E(\eta)$. We call $\xi^\perp$ the orthogonal complement of ξ in η.

<u>Theorem</u>. $\xi^\perp$ is a vector bundle. $\xi^\perp \oplus \xi = \eta$.

<u>Proof.</u> We need only check $\xi^\perp$ is locally trivial. So let $b \in B$, and pick neighborhood $\mathcal{U}$ on which η and ξ are locally trivial.

We can choose orthonormal cross-sections $s_1, \dots, s_m$ for ξ, $s'_1, \dots, s'_n$ for η. Rewriting the s_i in the basis of the s'_i, we have an $m \times n$ matrix $A = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ which has rank m, in a neigh. of b, since the s_i are lin. independent.

This means that K columns of A are lin. indep, by reordering, they may as well be first K.

Claim. $s_1, \dots, s_m, s'_{m+1}, \dots, s'_n$ are lin. indep.

If not, ~~since~~ ~~$s_1, \dots, s_m$~~ something nonzero in $\mathrm{span}(s_1, \dots, s_m)$ is also in $\mathrm{span}(s'_{m+1}, \dots, s'_n)$. So something nonzero in $\mathrm{span}(s_1, \dots, s_m)$ is orthogonal to $s'_1, \dots, s'_m$. But then some nonzero combination of rows in the first K columns of A is 0, contradicting assertion that this has rank K.

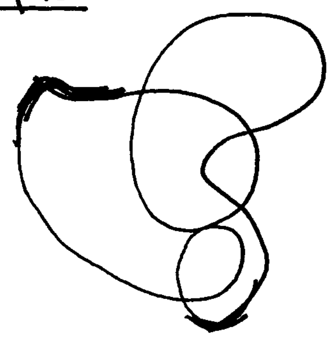
Now apply Gram-Schmidt to obtain orthomormal cross-sections $s_1, \dots, s_m, s_{m+1}, \dots s_n$. The cross sections $s_{m+1}, \dots, s_n$ ~~are~~ trivialize ξ^+ near b.

Example. $M \subset N$ are smooth manifolds, N Riemannian. $TM \subset TN$, the normal bundle of M in N is $(TM)^\perp = \nu$ and

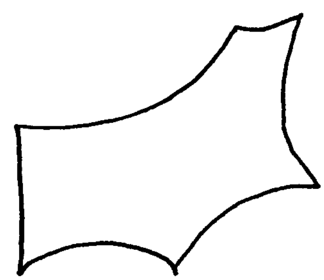
$$TM \oplus \nu \cong T_N|_M.$$

Definition. $f: M \to N$ is an immersion of M in N if $Df_x : TM_x \to TN_{f(x)}$ is injective for all $x \in M$.

Example.



immersion of $S^1 \hookrightarrow \mathbb{R}^2$



not an immersion.

Lemma. For any immersion $f: M \to N$ with N Riemannian, there is a Whitney sum

$$f^* T_{\S} N \cong TM \oplus \nu_f .$$

We call ν_f the normal bundle of the immersion f.

Proof. Obvious, but be sure to mention this TM is really an isomorphic copy of TM.

<u>Definition</u>. A continuous functor $T: \mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{V}$ is an operation so that

1. Given $V_1, \dots V_n \in \mathcal{V}$, we have $T(V_1, \dots, V_n) \in \mathcal{V}$.

2. If we have isomorphisms, $f_i : V_i \to W_i$, then we have an isomorphism
$$T(f_i) : T(V_i) \to T(W_i).$$
so
$$T(Id) = Id, \quad T(f_i \circ g_i) = T(f_i) \circ T(g_i).$$

3. $T(f_i)$ depends continuously on f_i (as matrices).

<u>Theorem</u>. Given a continuous functor T of n vector spaces and n vector bundles $\xi_1, \dots, \xi_n$ over B, $\exists$ a vector bundle
$$T(\xi_1, \dots, \xi_n) \text{ over } B$$

with ~~total~~ ~~space~~ fibers

$$F_b = T(F_b(\xi_1), \ldots, F_b(\xi_n))$$

and total space given by the union of these fibers with a canonical topology induced by the topologies on the $E(\xi_i)$.

(see Theorem 3.6 in your book).

Problems. 3A.