Pontrjagin classes

Given any real vector space V we can complexify by tensoring on a $\mathbb{C}$ to get a complex vector space $V \otimes \mathbb{C}$.

Given a real vector bundle ξ we can then build a complex vector bundle $\xi \otimes \mathbb{C}$ by tensoring each fiber with $\mathbb{C}$.

Suppose we have some fiber F of ξ and

$$\vec{v} \in F \otimes \mathbb{C}.$$

Then $\vec{v}$ is an equivalence class of elements in the form $\vec{u} \otimes z$ where $\vec{u} \in F$, $z \in \mathbb{C}$.
But

$$\begin{aligned}
\vec{u} \otimes z = \vec{u} \otimes (x + iy) &= \vec{u} \otimes x + \vec{u} \otimes iy \\
&= x\vec{u} \otimes 1 + y\vec{u} \otimes i \\
&= \vec{v}_1 + i\vec{v}_2, \quad \vec{v}_1, \vec{v}_2 \in F.
\end{aligned}$$

So we can see (since $\vec{v}_1, \vec{v}_2$ unique) that

$$F \otimes \mathbb{C} \cong F \oplus iF$$

isomorphic as vector spaces

so

isomorphic as Real vector bundles

$$(\xi \otimes \mathbb{C})_{R} \cong \xi \oplus \xi$$

with complex structure $J(x,y) = (-y,x)$.

Lemma. $\xi \otimes \mathbb{C} \cong \overline{\xi \otimes \mathbb{C}}$.

Proof. Consider the map $f : x+iy \mapsto x-iy$. This takes fibers to fibers, the total space homeomorphically to the total space, is $\mathbb{R}$-linear on each fiber. We must check it is $\mathbb{C}$-conjugate linear.

$$\begin{aligned} f(i(x+iy)) &= f(-y+ix) = -y-ix \\ &= -i(x-iy) \\ &= -i\, f(x+iy). \end{aligned}$$

Now $\xi \otimes \mathbb{C}$ is a complex bundle so we can take the Chern class

$$c(\xi \otimes \mathbb{C}) = 1 + c_1(\xi \otimes \mathbb{C}) + \dots + c_n(\xi \otimes \mathbb{C}).$$

$$\parallel$$

$$c(\overline{\xi \otimes \mathbb{C}}) = 1 - c_1(\xi \otimes \mathbb{C}) + \dots \pm c_n(\xi \otimes \mathbb{C}).$$

We see that the odd Chern classes must be torsion elements of order 2.

<u>Definition</u>. The ith Pontrjagin class

$$p_i(\xi) \in H^{4i}(B; \mathbb{Z}) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}).$$

Now the total Pontrjagin class is

$$p(\xi) = 1 + p_1(\xi) + \dots + p_{\lfloor n/2 \rfloor}(\xi)$$

where $\lfloor n/2 \rfloor$ = largest integer $\leq n/2$.

It is clear that

Lemma. Pontrjagin classes are natural w.r.t. bundle maps and $p(\xi \oplus \varepsilon^k) = p(\xi)$.

The product formula looks funny:

Theorem. $p(\xi \oplus \eta) \equiv p(\xi)\, p(\eta)$ mod elts of order 2.

or $2(p(\xi \oplus \eta) - p(\xi)p(\eta)) = 0$.

Proof. Observe

$$(\xi \oplus \eta) \otimes \mathbb{C} \cong (\xi \otimes \mathbb{C}) \oplus (\xi \otimes \mathbb{C}).$$

So

$$c_k((\xi \oplus \eta) \otimes \mathbb{C}) = c_k((\xi \otimes \mathbb{C}) \oplus (\eta \otimes \mathbb{C}))$$

$$= \sum_{i+j=k} c_i(\xi \otimes \mathbb{C})\, c_j(\eta \otimes \mathbb{C})$$

The odd elements are all of order 2, so

$$c_{2k}((\xi \oplus \eta)\otimes \mathbb{C}) \overset{\text{mod elts of order 2}}{\equiv} \sum_{i+j=k} c_{2i}(\xi\otimes\mathbb{C})\, c_{2j}(\eta\otimes\mathbb{C})$$

Do the signs work out? Well, & $i+j=k$, so $(-1)^k = (-1)^i(-1)^j$. Thus

$$(-1)^k\left(c_{2k}((\xi\oplus\eta)\otimes\mathbb{C})\right) \overset{\text{mod elts of order 2}}{\equiv} \sum_{i+j} (-1)^i c_{2i}(\xi\otimes\mathbb{C})\,(-1)^j c_{2j}(\eta\otimes\mathbb{C})$$

as required. $\square$.

Example. Since $T(S^n)\oplus N(S^n) = T(\mathbb{R}^{n+1})|_{S^n} = \varepsilon^{n+1}$, and $N(S^n) = \varepsilon^1$, we have

$$p(S^n) = 1, \text{ by previous lemma.}$$

To get a better example, we step back and consider Pontrjagin vs. Chern.

We have

$$\begin{tikzcd}
& \text{real } n\text{-bundle } \xi \arrow[dr, bend left, "\otimes \mathbb{C}"] & \\
\begin{matrix}\text{oriented real} \\ 2n\text{-bundle} \\ (\xi \times \mathbb{C})_{\mathbb{R}}\end{matrix} \arrow[ur, bend left, "\begin{matrix}\text{forget orientation} \\ \text{to get} \\ (\xi\otimes\mathbb{C})_{\mathbb{R}} \cong \xi\oplus\xi\end{matrix}"] & & \text{complex } n\text{-bundle } \xi\otimes\mathbb{C} \arrow[ll, bend left, "\begin{matrix}\text{take underlying} \\ \text{real bundle}\end{matrix}"]
\end{tikzcd}$$

This circle (helix?) starts with a bundle and gives you a new bundle of twice the dimension. We could start with a complex bundle ω, too...

Lemma. $\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}$.

Proof. We saw that any real vector space V has

$$V \otimes \mathbb{C} \cong V \oplus V \text{ with complex structure } J(x,y) = (-y,x).$$

Now suppose $V = F_{\mathbb{R}}$ where F is the fiber of a complex vector bundle ω.

We claim

$$g(\vec{x}) = (\vec{x}, -i\vec{x}) \text{ from } F \to V \oplus V$$

is complex linear. This <u>means</u>

$$g(i\vec{x}) = (i\vec{x}, -\vec{x}) = (-(-i\vec{x}), \vec{x}) = J(\vec{x}, -i\vec{x})$$
$$= Jg(\vec{x}).$$

Further, we claim

$$h(\vec{x}) = (\vec{x}, i\vec{x})$$

is conjugate-linear, or

$$h(i\vec{x}) = (i\vec{x}, -\vec{x}) = -(-i\vec{x}, \vec{x})$$
$$= -J(\vec{x}, i\vec{x})$$
$$= -Jh(\vec{x}).$$

Now consider any element $(x,y) \in V \oplus V$.
We claim that

$$(x,y) = g\left(\frac{x+iy}{2}\right) + h\left(\frac{x-iy}{2}\right)$$

$$= \left(\frac{x+iy}{2}, \frac{-ix+y}{2}\right) + \left(\frac{x-iy}{2}, \frac{ix+y}{2}\right)$$

$$= (x,y).$$

So the images $g(F)$ and $h(F)$ have $V \oplus V = g(F) \oplus h(F)$. Further, if we ~~take h to be a~~ compose h with ~~a~~ the identity map from F to $\overline{F}$ (which is also conjugate-linear), this shows

$$V \oplus U \cong F \oplus \overline{F}$$

canonical isomorphism of complex vector spaces.

so $F_{\mathbb{R}} \otimes \mathbb{C} \cong F \oplus \overline{F}$, and the rest follows. $\square$

Corollary. For any complex n-plane bundle ω, $c_i(\omega)$ determine $p_K(\omega_R)$ by

$$1 - p_1 + p_2 - \cdots \pm p_n = (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + c_2 \cdots + c_n).$$

This implies

$$p_K(\omega_R) = c_K(\omega)^2 - 2\, c_{K-1}(\omega)\, c_{K+1}(\omega) + \cdots \pm 2\, c_{2K}(\omega).$$

Proof. By previous,

$$p_K(\omega_R) = (-1)^K c_{2K}(\omega_R \otimes \mathbb{C})$$

$$= (-1)^K c_{2K}(\omega \oplus \bar{\omega})$$

$$= (-1)^K \sum_{i+j=2K} c_i(\omega)\, c_j(\bar{\omega})$$

$$= (-1)^K \sum_{i+j=2K} c_i(\omega)\, (-1)^j c_j(\omega). \qquad \square$$

We now compute

$$p_k\left((T(\mathbb{C}P^n))_{\mathbb{R}}\right) \text{ - Pontrjagin classes of real bundle underlying tangent bundle of } \mathbb{C}P^n.$$

We see

$$(1-p_1+\dots\pm p_n) = (1-c_1+\dots\pm c_n)(1+c_1+\dots+c_n)$$
$$= (1-a)^{n+1}(1+a)^{n+1}$$
$$= (1-a^2)^{n+1}$$

so

$$p = 1+p_1+\dots+p_n = (1+a^2)^{n+1}$$

or

$$p_k(T(\mathbb{C}P^n)_{\mathbb{R}}) = \binom{n+1}{k} a^{2k},$$

for $1 \leq k \leq n/2$.

This lets us compute some Pontrjagin classes and show they are nonzero.

$\vdots$

$$p(\mathbb{C}P^6) = 1 + 7a^2 + 21a^4 + 35a^6.$$

Now consider our helix yet again

$$\begin{tikzcd} & \text{real} \arrow[dr, bend left] & \\ \text{oriented} \arrow[ur, bend left] & & \text{complex} \arrow[ll, bend left] \end{tikzcd}$$

Suppose the intial bundle was oriented.

Lemma. $(\xi \otimes \mathbb{C})_R \cong \xi \oplus \xi$ under an orientation preserving isomorphism if $n(n-1)/2$ is even and an or. reversing isomorph. if $n(n-1)/2$ is odd.

Proof. Let $v_1, \ldots, v_n$ be an ordered basis for a fiber F of ξ. The orientation on $F \otimes \mathbb{C}_\mathbb{R}$ determined by the complex structure is given by basis $v_1, iv_1, \ldots, v_n, iv_n$. The ~~to~~ orientation on $F \oplus F \cong (F \otimes \mathbb{C})_\mathbb{R}$ determined by orient. on F is given by $v_1, \ldots, v_n, iv_1, \ldots, iv_n$. To switch from one to the other takes

$$(n-1) + (n-2) + \ldots + 1 = n(n-1)/2 \text{ swaps. } \square$$

Corollary. If ξ is an oriented $2k$-bundle,

$$p_k(\xi) = e(\xi)^2.$$

Cohomology of $\tilde{G}_n(\mathbb{R}^\infty)$ ~~again~~

If we choose a ring of coefficients Λ containing $1/2$ (with no 2-torsion) such as $\mathbb{Z}[1/2]$, we can compute cohomology of <u>oriented</u> n-planes in $\mathbb{R}^\infty$, $\tilde{G}_n(\mathbb{R}^\infty)$

Theorem.

$H^*(\tilde{G}_{2m+1}, \Lambda)$ is a polynomial ring generated by $p_1(\gamma^{2m+1}), \ldots, p_m(\gamma^{2m+1})$

$H^*(\tilde{G}_{2m}, \Lambda)$ is a polynomial ring generated by $p_1(\gamma^{2m}), \ldots, p_m(\gamma^{2m})$ and the Euler class $e(\gamma^{2m})$.

These generators have the relations $e = 0$, for $\tilde{G}_{2m+1}$, $e^2 = p_m$ for $\tilde{G}_{2m}$

Proof (by induction on n, for G_n)

$$\tilde{G}_1(\mathbb{R}^N) \cong S^{N-1},$$

so $\tilde{G}_1$ has no cohomology.

Suppose we know theorem for $n-1$. As before, we can build a sequence

$$\to H^i(\tilde{G}_n) \xrightarrow{\cup e} H^{i+n}(\tilde{G}_n) \xrightarrow{\lambda} H^{i+n}(\tilde{G}_{n-1}) \to H^{i+1}(\tilde{G}_n) \to$$

using the Gysin sequence of $\tilde{\gamma}^n$, where e is the Euler class of $\tilde{\gamma}^n$ and $\lambda = (f^*)^{-1}\pi_0^*$, where $f : \tilde{E}_0(\tilde{\gamma}^n) \to \tilde{G}_{n-1}$ is the "perp map". As before,

$$\lambda(p_i(\tilde{\gamma}^n)) = p_i(\tilde{\gamma}^{n-1})$$

(since this is true for Chern classes, and the Pontrjagin classes <u>are</u> Chern classes.)

Case 1. If n is even, this is just like our calculation of ~~$H^*(G_n,\mathbb{Z})$~~ $H^*(G_n(\mathbb{C}^\infty),\mathbb{Z})$. λ is surjective by induction, so the sequence becomes

$$0 \longrightarrow H^i(\tilde{G}_n) \xrightarrow{\ ue\ } H^{i+n}(\tilde{G}_n) \xrightarrow{\lambda} H^{i+n}(\tilde{G}_{n-1}) \to 0$$

and we know $\tilde{G}_{n-1}$ has cohomology generated by $p_1, \ldots, p_{(n/2)-1}$. Now we show $H^{i+n}(\tilde{G}_n)$ is generated by Pontrjagin classes by induction on i as before.

Case 2. If n is odd, there's no Euler class, so the sequence is

$$0 \longrightarrow H^j(\tilde{G}_{2m+1}) \xrightarrow{\ \lambda\ } H^j(\tilde{G}_{2m}) \to H^{j-2m}(\tilde{G}_{2m+1}) \to 0$$

Now the first map λ is injective, so $H^*(\tilde{G}_{2m+1}) \subset H^*(\tilde{G}_{2m})$. Let A^* be the polynomial algebra $\Lambda[p_1, \ldots, p_m] \subset H^*(G_{2m})$. We know that

$$A^* \subset \lambda(H^*(\tilde{G}_{2m+1}))$$

since λ maps p_i's to p_i's and $\tilde{G}_{2m+1}$ <u>has</u> Pontrjagin classes $p_1, \ldots, p_m$ (we don't know whether $H^*(\tilde{G}_{2m+1})$ has anything <u>else</u>). We claim $A^* = \lambda(H^*(\tilde{G}_{2m+1}))$, which would complete proof.

Now $A^* \subset \lambda(H^*(\tilde{G}_{2m+1})) \Rightarrow \forall j$ we have

$$\operatorname{rank} A^j \leq \operatorname{rank} H^j(\tilde{G}_{2m+1})$$

(where <u>rank</u> = max # of Λ-lin. indep. elts)

By induction each $x \in H^j(\tilde{G}_{2m})$ has

$$x = a + ea', \quad a \in A^j, \; a' \in A^{j-2m}$$

because we have

$$0 \longrightarrow H^{j-2m}(\tilde{G}_{2m}) \xrightarrow{ve} H^j(\tilde{G}_{2m}) \xrightarrow{\lambda} H^j(\tilde{G}_{2m-1}) \to 0$$

and by induction, we know that ~~every~~

$$H^{j-2m}(\tilde{G}_{2m}) = A^{j-2m}, \quad H^j(\tilde{G}_{2m-1}) = A^j.$$

Further, this is unique, since ve is injective. So we have

$$H^j(\tilde{G}_{2m}) = A^j \oplus A^{j-2m}$$

and

$$\operatorname{rank} H^j(\tilde{G}_{2m}) = \operatorname{rank} A^j + \operatorname{rank} A^{j-2m}.$$

But using the short exact sequence above, we have

$$\operatorname{rank} H^j(\hat{\tilde{G}}_{2m}) = \operatorname{rank} H^j(\hat{\tilde{G}}_{2m+1}) + \operatorname{rank} H^{j-2m}(\hat{\tilde{G}}_{2m+1})$$

So

$$\operatorname{rank} A^j + \operatorname{rank} A^{j-2m} = \operatorname{rank} H^j(\hat{\tilde{G}}_{2m+1}) + \operatorname{rank} H^{j-2m}(\hat{\tilde{G}}_{2m+1}),$$

but since each summand on lhs $\leq$ corresponding summand on rhs, we must have

$$\operatorname{rank} A^j = \operatorname{rank} H^j(\hat{\tilde{G}}_{2m+1})$$

If these were vector spaces, we'd be done, but they are Λ-modules. Still, if $A^j \neq \lambda(H^j(\hat{\tilde{G}}_{2m+1}))$, then $\exists$

$$a + ea' \in \lambda(H^j(\hat{\tilde{G}}_{2m+1})), \text{ with } a' \neq 0$$

since $\lambda(H^j(\tilde{G}_{2m+j})) \subset H^j(G_{2m}) = A^j \oplus A^{j-2m}$.

Now by the same decomposition, this element could not satisfy a linear dependence with elts of A^j, so rank $H^j(\tilde{G}_{2m+1}) >$ rank A^j.

Partitions, Chern numbers, $\mathbb{C}P^n$ example:

classifying space approach to $H^{2n}(G_n(\mathbb{C}^\infty);\mathbb{Z})$

Pontrjagin #s.

More applications of Pontrjagin

We recall (from a while back!) that

Theorem. If B is a smooth compact $(n+1)$-manifold with boundary M, then M has no SW numbers.

In fact, the same proof tells us that M has no Pontrjagin numbers. So

Example. $\mathbb{C}P^{2n}$ is not an oriented boundary. Even $\mathbb{C}P^{2n} \sqcup \mathbb{C}P^{2n}$ is not an oriented boundary, which is bizarre when you think about it, because $\mathbb{C}P^{2n} \sqcup -\mathbb{C}P^{2n}$ is the boundary of $\mathbb{C}P^{2n} \times I$.

Now we find a new basis for the homogenous symmetric polys of degree K (denoted S^K). (3)

Definition. Two monomials in $t_1, \dots, t_n$ are equivalent $\Leftrightarrow$ they are related by a permutation of $t_1, \dots, t_n$. Let

$$\sum_e t_1^{a_1} \cdots t_r^{a_r} = \text{summation of all monomials equivalent to } t_1^{a_1} \cdots t_r^{a_r}$$

Example. $\sigma_K = \sum_e t_1 \cdots t_K$.

Lemma. An additive basis for S^K is given by $\sum_e t_1^{a_1} \cdots t_r^{a_r}$ where $a_1, \dots, a_r$ ranges over all partitions of K with length $r \leq n$.

Now we can do something clever. (4)
Given a partition $I = i_1, \dots, i_r$ of K,
we can define s_I as follows.

Choose $n \geq K$, and let $\sigma_1, \dots, \sigma_K$ be the first K elementary symmetric polynomials in n variables and let

$$s_I(\sigma_1, \dots \sigma_K) = \sum_e t^{i_1} \dots t^{i_r}$$

Since the polynomial on the right is surely symmetric, and the symmetric polynomials are a polynomial algebra with no relations over the σ_i, s_I is a uniquely defined polynomial in K variables.

It is clear that there are partitions of K such polynomials S_I, and that they are linearly independant and form a basis for S^K.

Examples

$S(\,) = 1$ $\quad K=0$

$S_1(\sigma_1) = \sigma_1$ $\quad K=1$

$S_2(\sigma_1,\sigma_2) = \sigma_1^2 - 2\sigma_2$

$S_{1,1}(\sigma_1,\sigma_2) = \sigma_2$

$K=2$

$S_3(\sigma_1,\sigma_2,\sigma_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$

$S_{1,2}(\sigma_1,\sigma_2,\sigma_3) = \sigma_1\sigma_2 - 3\sigma_3$

$S_{1,1,1}(\sigma_1,\sigma_2,\sigma_3) = \sigma_3$

$K=3$

and so forth.

Now if an n-plane bundle ω splits as a Whitney sum of n line bundles $\eta_1 \oplus \cdots \oplus \eta_n$, then

$$1 + c_1(\omega) + \cdots + c_n(\omega) = (1 + c_1(\eta_1)) \cdots (1 + c_1(\eta_n))$$

shows that

$$c_k(\omega) = \sigma_k(c_1(\eta_1), \ldots, c_1(\eta_n)).$$

Example. Let $\gamma^1 \times \cdots \times \gamma^1$ be the n-fold product of line bundles over $\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$. We know $H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) = \mathbb{Z}[a_1, \ldots, a_n]$ where the a_i have degree 2 and we know

$$c(\gamma^1 \times \cdots \times \gamma^1) = (1 + a_1) \cdots (1 + a_n).$$

Now we know that the Chern classes of $\gamma^1 \times \cdots \times \gamma^1$ are n algebraically independent symmetric polynomials in the a_i.

Now these Chern classes are pullbacks of the Chern classes of γ^n over $G_n(\mathbb{C}^\infty)$. (7)

Since $H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) = \mathbb{Z}[c_1(\gamma^n),\dots,c_n(\gamma^n)]$ is the same size as $\mathbb{Z}[a_1,\dots,a_n]$, we must be mapping isomorphically from one to the other.

Now that means the set

S_I(chern classes of $\gamma^1 \times \dots \times \gamma^1$)

= S_I(symmetric polynomials in a_i)

= monomial generators of degree K for $\mathbb{Z}[a_1,\dots,a_n]$

= a basis for degree K part of $\mathbb{Z}[a_1,\dots,a_n]$

but under the isomorphism, this means

S_I(chern classes of γ^n)

= a basis for degree K part of $\mathbb{Z}[c_1(\gamma^n), \ldots, c_n(\gamma^n)]$

= a basis for $H^{2K}(G_n(\mathbb{C}^\infty);\mathbb{Z})$.

So we have come up with a new and useful basis for $H^{2K}(G_n(\mathbb{C}^\infty);\mathbb{Z})$.

Let ω be an n-plane bundle, with base B and Chern class $c = 1 + c_1 + \dots + c_n$

For any partition I of k,

$$s_I(c_1, \dots, c_n) \in H^{2k}(B; \mathbb{Z})$$

will be denoted s_I, or $s_I(c(\omega))$.

Lemma.

$$s_I(c(\omega \oplus \omega)) = \sum_{JK = I} s_J(c(\omega))\, s_K(c(\omega'))$$

Here two partitions $J = j_1, \dots, j_r$ and $K = k_1, \dots, k_s$ of j and k are juxtaposed to make a partition of $j+k$ by writing

$$JK = j_1, \dots, j_r, k_1, \dots, k_s$$

Example. $s_{\{k\}}(c(\omega \oplus \omega')) = s_{\{k\}}(c(\omega)) + s_{\{k\}}(c(\omega'))$

For each partition I of n, given a complex n-manifold K^n, let

$$s_I(K^n) = \langle s_I(c(TK^n)), \mu_{2n} \rangle \in \mathbb{Z}.$$

This is a certain linear combination of Chern numbers.

Corollary. $$s_I(K^m \times L^n) = \sum_{I_1 I_2 = I} s_{I_1}(K^m) s_{I_2}(L^n).$$

Proof. $T(K^m \times L^n) \cong (\pi_1^* TK^m) \oplus (\pi_2^* TL^n)$ where π_1 and π_2 are projections to K and L respectively. So

$$s_I(K^m \times L^n) = \sum_{I_1 I_2 = I} \langle s_{I_1}(TK^m) \times s_{I_2}(TK^n), \mu_{2(m+n)}(K^n \times L^m) \rangle$$

$$= \sum_{I_1 I_2 = I} \langle s_{I_1}(TK^m), \mu_{2m}(K^m) \rangle \langle s_{I_2}(TL^n), \mu_{2n}(L^n) \rangle$$

$$= \sum_{I_1 I_2 = I} s_I(K^m)\, s_I(K^n).$$

Now for the single element partition $\{m+n\}$ of $m+n$, this is not a juxtaposition of other partitions, so

Corollary. For any product manifold, $K^m \times L^n$, we have $s_{m+n}[K^m \times L^n] = 0$.

Example. Consider $\mathbb{C}P^n$. We have

$$c(\mathbb{C}P^n) = (1+a)^{n+1},$$ so

$$c_k(\mathbb{C}P^n) = \sigma_k(\underbrace{a, \dots, a}_{n+1 \text{ times}})$$

so

$$\cancel{s_K(\sigma_1, \dots, \sigma_K) =}$$

~~function of~~ symmetric polynomials in $c_1, \dots$

We now write Pontrjagin numbers and Chern numbers a different way using some algebra tricks.

Definition A polynomial $f(t_1, \ldots, t_n)$ is <u>symmetric</u> if it is invariant under any permutation of the t_i. The symmetric polynomials are generated by the <u>elementary symmetric polynomials</u>. ~~of degree k, denoted~~ where σ_k is the unique elementary symmetric polynomial of degree k and

$$1 + \sigma_1 + \ldots + \sigma_n = (1+t_1)(1+t_2)\cdots(1+t_n).$$

Theorem. A basis for the homogenous symmetric polynomials of degree k is given by products

$$\sigma_{i_1} \cdots \sigma_{i_r}, \quad i_1, \ldots, i_r \text{ is a partition of } k \text{ with each element} < n$$

so

$$s_k(c_1, \dots, c_k) = s_k(\sigma_1, \dots, \sigma_k)$$

$$= \sum_{a \in \underbrace{\{a, \dots, a\}}_{n+1 \text{ times}}} a^k = (n+1)a^k$$

so

$$s_n(c_1, \dots, c_n) = (n+1)a^n$$

so

$$s_n(\mathbb{C}P^n) = \langle (n+1)a^n, \mu_{2n} \rangle = n+1 \neq 0.$$

Thus $\mathbb{C}P^n$ is not a product!