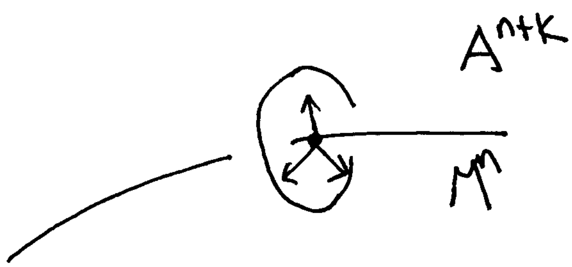
II. Characteristic class computations in a smooth manifold.

We are going to skip the proof of the Thom Isomorphism Theorem. It is long and technical and boils down to: works on the chain level for a trivial bundle— now extend.

It is more fun to try to bring home some cool information about manifold topology for smooth manifolds.

The Normal Bundle



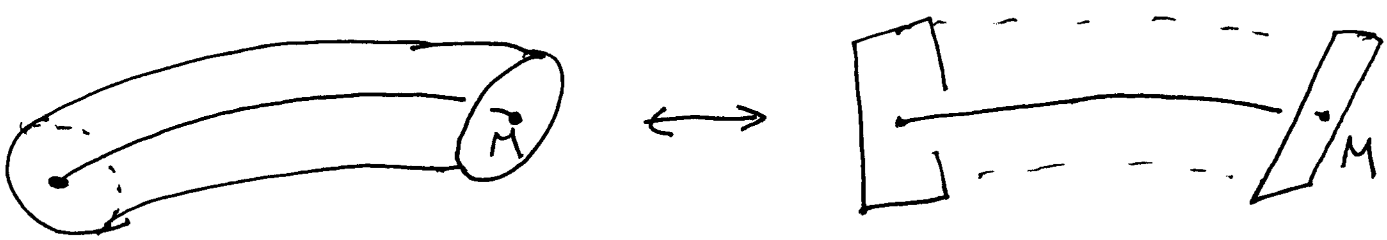
Consider

M^n embedded in A^{n+k}

We want to study smooth embeddings by understanding characteristic classes in their normal bundles.

Tubular Neighborhood Theorem.

There is an open neighborhood of M in A which is diffeomorphic to the total space of the normal bundle of M under a diffeo. that maps $x \in M$ to $(x,0) \in NM$.

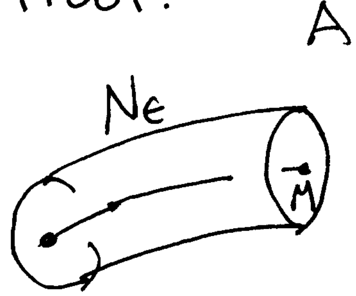


Proof. Use exp to define a local map for compact M. By various ODE theorems, this map is smooth near $M \times 0 \subset E(\nu(M))$. By inverse function theorem, there is then a local diffeomorphism.

Now we have (in any coefficient ring R)

Corollary. If M is closed in A, the cohomology ring $H^*(E, E_0; R)$ from NM_ε is canonically isomorphic to $H^*(A, A-M; R)$.

Proof.



Let N_ε be the tubular neighborhood of M from the theorem. By excision, there is a canonical isomorphism

$$H^*(A, A-M) \to H^*(N_\varepsilon, N_\varepsilon - M)$$

Now there is a ~~n~~ set $E(\varepsilon)$ in $E(NM)$ which ~~maps~~ is diffeomorphic (by Exp) to N_ε. We take

$$\mathrm{Exp}: (E(\varepsilon), E(\varepsilon)_0) \to (N_\varepsilon, N_\varepsilon - M)$$

to induce a corresponding Exp^* on cohomology. We then have excision

$$H^*(E(\varepsilon), E(\varepsilon)_0) \cong H^*(E, E_0). \quad \square$$

We know by Thom theorem, $\exists$ a fundamental class $u \in H^k(E, E_0; \mathbb{Z}/2)$. This corresponds to $u' \in H^k(A, A-M; \mathbb{Z}/2)$.

If NM is orientable, an orientation gives a fundamental class $u \in H^k(E, E_0; \mathbb{Z})$ $\cong u' \in H^k(A, A-M; \mathbb{Z})$.

Theorem. If M embeds in A as a closed subset, then the inclusions $M \hookrightarrow A \hookrightarrow A, A-M$ induce restriction homomorphisms

$$H^k(A, A-M) \to H^k(A) \to H^k(M)$$

have the following properties.

i) in $\mathbb{Z}/2$ coefficients, u' (the fund. class) maps to $w_k(NM)$, the top SW class.

ii) in $\mathbb{Z}$ coefficients (if NM oriented), u' maps to Euler class $e(NM)$.

Proof. Let

$$s : M \to E(NM)$$

be the 0-section of NM. It induces a canonical isomorphism $s^*: H^*(E) \to H^*(M)$. We ~~[illegible]~~ claim that

$$H^k(E, E_0) \to H^k(E) \xrightarrow{s^*} H^k(M)$$

maps the fundamental class u to the top SW class $\omega_k(NM)$. To see this, we compute the image of $s^*(U|_E)$ under the Thom isomorphism

$$\varphi : H^k(M) \to H^{2k}(E, E_0).$$

We see

$$\begin{aligned} \varphi(s^*(U|_E)) &= \pi^*(s^*(U|_E)) \cup u \\ &= U|_E \cup u \\ &= u \cup u = Sq^k(u). \end{aligned}$$

So

$$S^*(u|_E) = \varphi^{-1} Sq^k(u) = \omega_k(NM).$$

Now the normal bundle NM is diffeomorphic to the ε-neighborhood N_ε, so this implies the same about the pairs $(N_\varepsilon, N_\varepsilon - M)$ — that is

$$H^k(N_\varepsilon, N_\varepsilon - M) \to H^k(N_\varepsilon) \to H^k(M)$$

maps u' to $\omega_k(NM)$. Now in the long exact sequence of $(A, A-M)$ and $(N_\varepsilon, N_\varepsilon - M)$ we have

$$\begin{tikzcd}
H^k(A, A-M) \arrow[r] \arrow[d, "\cong"] & H^k(A) \arrow[d] \\
H^k(N_\varepsilon, N_\varepsilon - M) \arrow[r] & H^k(M)
\end{tikzcd}$$

this commutes with the restriction homomorphs. from $N_\varepsilon \hookrightarrow A$, $M \hookrightarrow A$ shown vertically

We now have a diagram chase.

On the bottom, we have seen

$$u' \mapsto w_k(\nu K)$$

But the left vertical arrow is an isomorphism by excision, (of $A - N_\varepsilon$) so we have

$$u' \in H^k(A, A-M) \to H^k(A) \to H^k(M) \ni w_k(\nu M).$$

We never used the $\mathbb{Z}/2$ hypothesis except to assume a fundamental class exists, so $\mathbb{Z}$ case is similar. $\square$

Definition. ~~If $M = M^n$ is smoothly embedded as a closed~~
The image of u' in $H^k(A)$ is called the dual class to $M^n \subset A^{n+k}$.

Note. If $u'|_A = 0$, then $w_k(\nu M)$ or $e(\nu M) = 0$.

This means

$\mathbb{R}P^{(2^n)}$ cannot be smoothly embedded as a closed subset of $\mathbb{R}^{2\cdot(2^n)-1}$

or the smallest $\mathbb{R}^M$ so that $\mathbb{R}P^{(2^n)}$ embeds (smoothly, as a closed subset) is $M = 2\cdot(2^n)$.

(We previously proved that $\mathbb{R}P^{(2^n)}$ does not immerse in anything smaller than $\mathbb{R}^{2(2^n)-1}$.)

Remark. The Mobius band (total space of γ_1^1) is claimed to have $\bar{w}_1(TM) \neq 0$ as a 2-manifold. (Why?) I think there's a general theorem here about $T(TM)$ of $T(E(\xi))$.

Similarly, Whitney immersion claims $\mathbb{R}P^2$ immerses in $\mathbb{R}^3$, but we have shown it can embed only in $\mathbb{R}^4$!

Corollary. If $M = M^n$ is smoothly embedded in $\mathbb{R}^{n+k}$ as a closed subset of $\mathbb{R}^{n+k}$, then $w_k(NM) = 0$. If everything is oriented, $e(NM) = 0$.

Proof. In $H^k(A, A - M) \to H^k(A) \to H^k(M)$, the middle group is zero. $\square$

Examples. We recall that since

$$TM \oplus NM = T\mathbb{R}^{n+k} = \text{trivial},$$

~~we~~ the class $w_k(NM)$ corresponds to an "inverse" class $\bar{w}_k(TM)$.

So for $M = \mathbb{R}P^{2^n}$, we have

$$\begin{aligned} w(\mathbb{R}P^{2^n}) &= 1 + a + a^{2^n} \\ &= 1 + a + a^{2^n}. \end{aligned}$$

and

$$\bar{w}(\mathbb{R}P^{2^n}) = 1 + a + a^2 + a^3 + \dots + a^{2^n - 1}$$

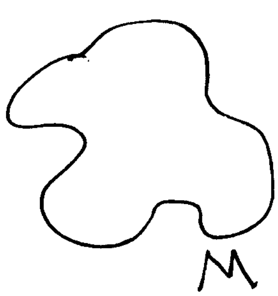
Characteristic class computations (continued)

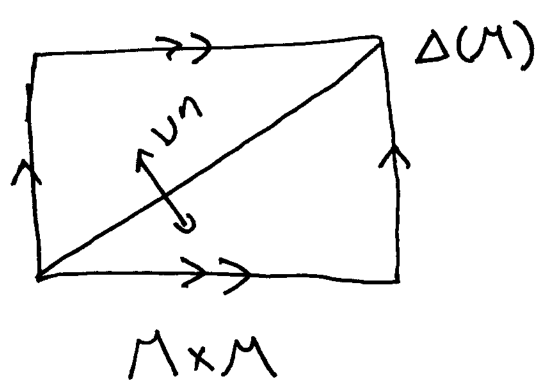
Our goal now is to understand the Stiefel-Whitney and Euler classes in a new and concrete way by studying the tangent bundle.

Suppose M is Riemannian. Then $M \times M$ is Riemannian if we assume that ~~the two copies of~~

$T_x M$ and $T_y M$ are orthogonal in $T_{(x,y)}(M \times M) = T_x M \oplus T_y M$.

Now consider $\Delta : M \to M \times M$,





Lemma. The normal bundle $N\Delta(M)$ of the diagonal embedding $\Delta(M)$ of $M \hookrightarrow M \times M$ is canonically isomorphic to TM.

Proof. We know that $T_{(x,x)}(M \times M) = T_xM \times T_xM$. Now the tangent space to $\Delta(M)$ is

$$(u,v) \in T_{(x,x)}\Delta(M) \iff u = v$$

so the normal space is

$$(u,v) \in N_{(x,x)}\Delta(M) \iff u = -v.$$

We can now map

$$(x,v) \longleftrightarrow ((x,x),(-v,v))$$

diffeomorphically from TM to $N\Delta(M)$. $\square$

Now it isn't surprising that

Lemma. An orientation for TM (as a bundle) gives rise to an orientation for M (as a manifold) and vice versa.

We now want to develop some general theory of $H^*(M)$, assuming M is oriented <u>or</u> our coefficients are $\mathbb{Z}/2\mathbb{Z}$.

We already saw

Cor. If M closed in A, $H^*(E,E_0)$ for the normal bundle of M in A is canonically isomorphic to $H^*(A, A-M)$.

In this case, we see $\exists$ a fundamental class

$$u' \in H^n(M \times M, M \times M - \Delta(M))$$

coming from the Thom class of $N\Delta(M)$. We proved that

$$\Delta^* u' \in H^n(M) = e(N\Delta(M)) = e(TM)$$

is the Euler class of M. (or the top SW class of M in $\mathbb{Z}/2$ coeffs).

We can say a bit more about this class. We know

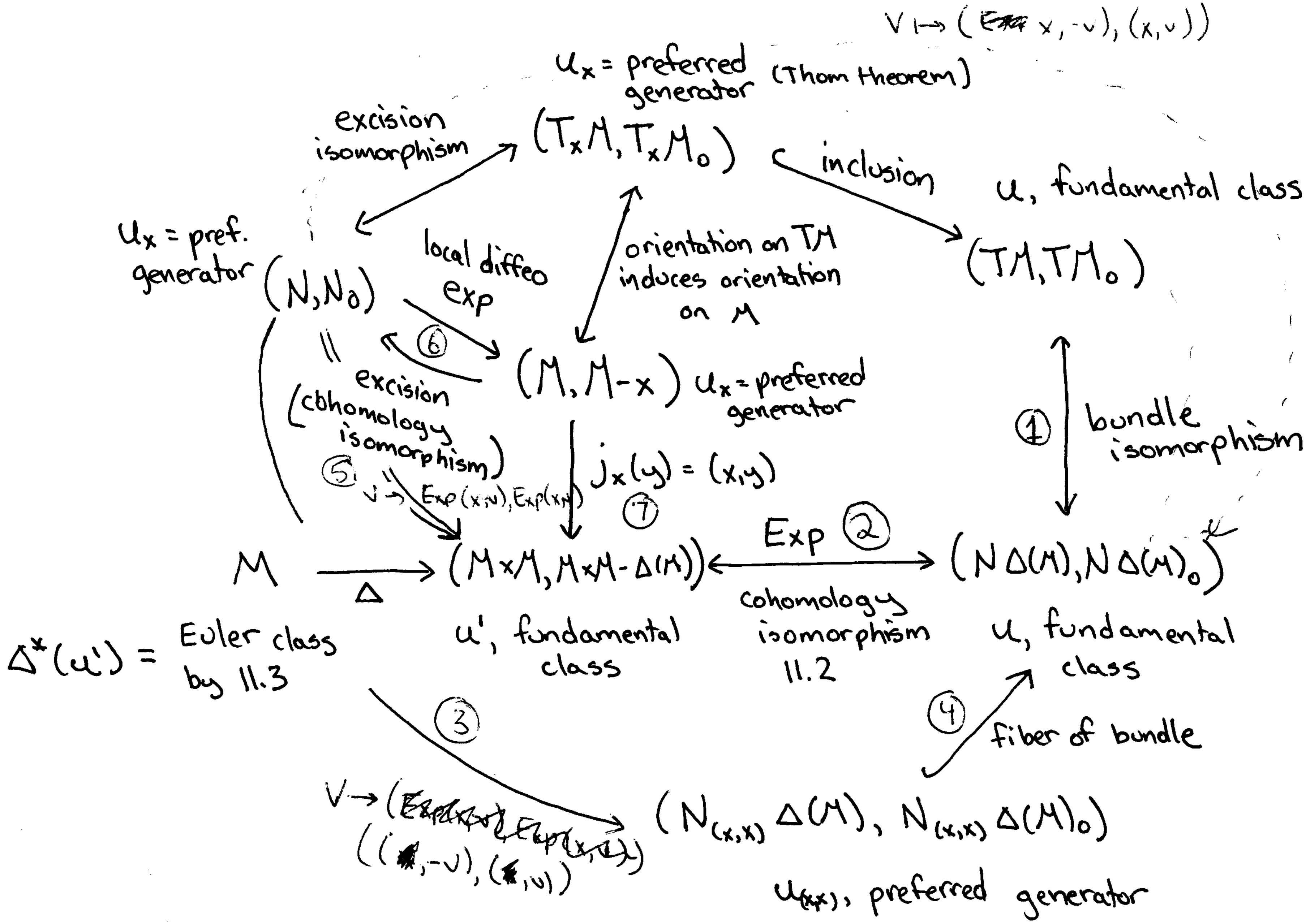
$$H^n(M, M-x) \text{ generated by } u_x \text{ s.t. } \langle u_x, \mu_x \rangle = 1$$

where μ_x comes from the orientation of M. Now we can embed

$$j_x : (M, M-x) \longrightarrow (M \times M, M \times M - \Delta(x))$$

by $j_x(y) = (x, y)$.

Let N = a neighborhood of 0 in $T_x M$.



Lemma 11.7. $u' \in H^n(M \times M, M \times M - \Delta(M))$ is uniquely characterised by the property that $j_x^*(u') = u_x$ for each $x \in M$.

Proof. Notice that (1) is a bundle isomorphism. So the fundamental class u of (TM, TM_0) maps to fundamental class u of $(N\Delta(M), N\Delta(M)_0)$. This is (uniquely), the class which restricts to the preferred generator in the fiber $(N_{(x,x)}\Delta(M), N_{(x,x)}\Delta(M)_0)$. Under the bundle isomorphism (1), this is the same as the fiber (T_xM, T_xM_0) under $v \mapsto (-v, v)$, and the preferred generators are the same, too.

Exp maps a neighborhood of the 0-section of $N\Delta(M)$ diffeomorphically onto a neighborhood of $\Delta(M)$ in arrow 2. This is a cohomology isomorphism for the pairs

$$(M\times M,\ M\times M-\Delta(M)) \longleftrightarrow (N\Delta(M), N\Delta(M)_0)$$

by II.3 (excision), mapping $u \leftrightarrow u'$.

The composition 3, 4, 2 then is arrow 5, so 5 pulls back u' to the preferred generator for (N, N_0).

Now the composition 6, 7 is $v \mapsto (x, \mathrm{Exp}(x,v))$. But 5 is $v \mapsto (\mathrm{Exp}(x,-v), \mathrm{Exp}(x,v))$ and this is clearly homotopic to 6, 7, by

$$v \mapsto (\mathrm{Exp}(x,-tv), \mathrm{Exp}(x,v)).$$

Thus 6, 7 maps u' to the generator for (N, N_0) and this defines u' uniquely. Yet 6 is an isomorphism, so this means 7 maps u' to the pref. generator for $(M, M-x)$ as desired. $\square$

(Whew!)

Now we observe that

$$M \times M \longrightarrow (M \times M,\ M \times M - \Delta(M))$$

so the restriction homomorphism maps u' to some $u'' \in H^n(M \times M)$ which we claim is "dual" to $\Delta(M)$. We call u'' the <u>diagonal</u> <u>cohomology</u> <u>class</u> in $M \times M$.

We claim first

Lemma. For any $a \in H^*(M)$,

$$(a \times 1) \cup u'' = (1 \times a) \cup u''.$$

Proof. Take a neighborhood N_ε of $\Delta(M)$. It is clear that N_ε deformation retracts to $\Delta(M)$, so

$$\begin{tikzcd} M \times M \arrow[r, "(x,y)\mapsto x", "p_1"'] \arrow[d, "p_2", "(x,y)\mapsto y"'] & M \\ M & \end{tikzcd}$$

So $p_1|_{N_\varepsilon}$ is homotopic to $p_2|_{N_\varepsilon}$. So

$$p_1^*(a) = a \times 1, \qquad p_2^*(a) = 1 \times a$$

have the same image under the restriction $H^i(M \times M) \to H^i(N_\varepsilon)$.

We now have

$$\begin{tikzcd} H^i(M\times M) \arrow[r, "\text{restriction}"] \arrow[d, "\cup u'"] & H^i(N_\varepsilon) \arrow[d, "\cup u'|_{(N_\varepsilon, N_\varepsilon - \Delta(M))}"] \\ H^{i+n}(M\times M, M\times M - \Delta(M)) \arrow[r, "\cong", "\text{excision}"'] & H^{i+n}(N_\varepsilon, N_\varepsilon - \Delta(M)) \end{tikzcd}$$

which commutes by naturality of cup.

We now ~~define~~ recall the slant product, which is meant to recall a sort of "cohomological division":

$$H^{p+q}(X\times Y) \otimes H_q(Y) \to H^p(X)$$

When coefficients are in a field,

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

by the Künneth formula. In this case, we can define a map

$$(H^*(X) \otimes H^*(Y)) \otimes H_*(Y) \to H^*(X)$$

by

$$a \otimes b \otimes \beta \mapsto a\langle b, \beta\rangle.$$

We denote this operation $p \otimes \beta \mapsto p/\beta$.

Claim. This operation is defined by

$$(a \times b)/\beta = a\langle b, \beta\rangle.$$

Now we observe

$$((a \times 1) \cup p)/\beta = a \cup (p/\beta).$$

To see this, we use the identity

$$(a \times b) \cup (c \times d) = \pm (a \cup c) \times (b \cup d)$$

We have $p = \sum k_{ij}\, c_i \times d_j$ since $p \in H^*(X \times Y)$.
So we have

$$(\alpha \times 1) \cup p = (a \times 1) \cup \left(\sum k_{ij}\, c_i \times d_j\right)$$

we see later (p19) this is +

$$= \overset{+}{k_{ij}} (a \cup c_i) \times (1 \cup d_j)$$

But $1 \cup d_j = d_j$, so this ~~is~~ gives

$$((a \times 1) \cup p)/\beta)$$

$$= \left(k_{ij} (a \cup c_i) \times d_j\right)/\beta$$

$$= k_{ij} (a \cup c_i) \langle d_j, \beta \rangle$$

$$= a \cup k_{ij}\, c_i \langle d_j, \beta \rangle$$

$$= a \cup (k_{ij}\, c_i \times d_j / \beta)$$

$$= a \cup (p/\beta). \qquad \square$$

Of course, we haven't proved this for coefficients not in a field. We point out that in this case the right thing to do is to define the same operation in <u>cellular</u> homology since the <u>cells</u> of $X \times Y$ are really cells $X \otimes$ cells Y so we can define things at the level of <u>cellular</u> chains and cochains.

We now go back to field coefficients and show:

Lemma. The diagonal class u'' and the fundamental or top class μ of $H_n(M)$ are related by $u''/\mu = 1 \in H^0(M)$. (when M is compact so μ is defined).

Proof. ~~P~~ Pick $x \in M$. We will compute the image of u''/μ under the restriction

$$H^0(M) \to H^0(x) = \Lambda \leftarrow \text{field of coeffs.}$$

Now slant is natural, so

$$\begin{tikzcd} H^n(M\times M) \arrow[r, "/\mu"] \arrow[d, "\text{restriction}"] & H^0(M) \arrow[d, "\text{restriction}"] \\ H^n(x\times M) \arrow[r, "/\mu"] & H^0(x) \end{tikzcd}$$

commutes. Now if

$$i_x : M \to M\times M \quad \text{maps } y \to (x,y)$$

then the left arrow is $1\times i_x^*$. So

$$1\times i_x^*(u'')/\mu = 1\,\langle i_x^*(u''),\mu\rangle.$$

Now the top class μ is the unique class so that

$$H_n(M) \to H_n(M, M-x)$$

maps μ onto the preferred generator u_x given by the orientation of M. Now

$u_x|_M = \mu$ by definition $\qquad\qquad$ $j_x^*(u') = u_x$ by last lemma

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, "i_x(y)=(x,y)"] & (M, M-x) \arrow[d, "j_x(y)=(x,y)"] \\
M \times M \arrow[r, hook] & (M\times M, M\times M - \Delta(M))
\end{tikzcd}$$

$u'' \leftarrow u'$, by definition of u''

commutes, so

$$i_x^*(u'') = j_x^*(u')|_M$$

and

$$\begin{aligned}
\langle i_x^*(u''), \mu \rangle &= \langle j_x^*(u')|_M, \mu \rangle \\
&= \langle j_x^*(u'), \mu_x \rangle \\
&= \langle u_x, \mu_x \rangle = 1.
\end{aligned}$$

But x was arbitrary, so u''/μ must be 1 in $H^0(M)$. $\square$

That's a lot of preliminaries! We study now cohomology of compact smooth, M with coeffs in field.
oriented

Poincare duality. To each basis $b_1, \ldots, b_r$ for $H^*(M)$ there is a dual basis $b_1^\#, \ldots, b_r^\#$ for $H^*(M)$ so that

$$\langle b_i \cup b_j^\#, \mu \rangle = \delta_{ij}$$

In these terms, the diagonal class

$$u'' = \sum_{i=1}^{r} (-1)^{\dim b_i} \, b_i \times b_i^\#.$$

We provide a slick proof of Poincare duality which also gives us our additional fact.

Proof. Our coefficients are in a field, so

$$H^*(M\times M) \cong H^*(M) \otimes H^*(M).$$

It follows that as $u'' \in H^n(M\times M)$,

$$u'' = b_1 \times c_1 + \dots + b_r \times c_r$$

where $c_1, \dots, c_r$ are classes with

$$\dim b_i + \dim c_i = n.$$

Now we know (Lemma 11.8),

$$(a\times 1) \cup u'' = (1\times a) \cup u''$$

so apply $/\mu$ to both sides. We get

$$\text{lhs} = (a\times 1) \cup u''/\mu$$
$$= \cancel{(a\times 1)}\ a \cup (u''/\mu) = a.$$

and

$$\text{rhs} = (1\times a) \cup u''/\mu$$
$$= (1\times a) \cup (\textstyle\sum b_j \times c_j)/\mu$$
$$= \pm \textstyle\sum (b_j \times (a \cup c_j))/\mu$$

$\pm$? what's the correct sign in

$$(a \times b) \cup (c \times d) = \pm (a \cup c) \times (b \times d) ?$$

well, we really view $\cup$ as the pullback of cross, so we have

$$(a \times b) \cup (c \times d) = \Delta^*((a \times b) \times (c \times d))$$

where $\Delta : X \times Y \to X \times Y \times X \times Y$. Now $(X \times Y) \times (X \times Y) = (X \times X) \times (Y \times Y)$ if we swap

$$\begin{aligned}(a \times b) \times (c \times d) &= a \times (b \times c) \times d \\ &= a \times (c \times b) \times d.\end{aligned}$$

Now cross product changes sign by $(-1)^{\dim b \dim c}$ when this happens, so

$$(a \times b) \cup (c \times d) = (-1)^{\dim b \dim c} (a \times c) \cup (b \times d).$$

so we have

$$\text{rhs} = \sum (-1)^{\dim a \dim b_j} \quad b_j \times (a \cup c_j)/\mu$$

$$= \sum (-1)^{\dim a \dim b_j} \quad b_j \langle a \cup c_j, \mu \rangle.$$

Now we have

$$a = \sum (-1)^{\dim a \dim b_j} \; b_j \langle a \cup c_j, \mu \rangle$$

for <u>any</u> a. Setting $a = b_i$, we see

$$b_i = \sum (-1)^{\dim b_i \dim b_j} \; b_j \langle b_i \cup c_j, \mu \rangle$$

we see that $\langle b_i \cup c_j, \mu \rangle = (-1)^{\dim b_i \dim b_j} \delta_{ij}$.

Now this means that $c_j = \pm b_i^{\#}$, where we can set the sign to $(-1)^{\dim b_i}$, since

$$(-1)^{(\dim b_i)^2} = (-1)^{\dim b_i}.$$

with this definition of the $b_i^{\#}$, we see Poincare duality and our extended fact at the same time. $\square$

Way cool!

We now connect Euler characteristic to the Euler class. We define

$$\chi(M) = \sum (-1)^k \operatorname{rank} H^k(M)$$
$$= \sum (-1)^k (\# \text{ of } k\text{-cells}).$$

~~Corollary~~

Theorem. $\langle e(TM), \mu \rangle = \chi(M)$ or
$\langle w_n(TM), \mu \rangle \equiv \chi(M) \bmod 2$,
depending on coefficients and orientation.

Proof. This is now easy!

We know that

$$\Delta^*(u'') = \text{Euler class of } N\Delta(M)$$
$$= \text{Euler class of } TM$$

So

$$e(TM) = \Delta^*\left(\sum (-1)^{\dim b_i}\, b_i \times b_i^{\#}\right)$$
$$= \sum (-1)^{\dim b_i}\, b_i \cup b_i^{\#}$$

Now if we evaluate each side on the top class μ of M, we get

$$\langle e(TM), \mu\rangle = \sum (-1)^{\dim b_i} \langle b_i \cup b_i^{\#}, \mu\rangle$$
$$= \sum (-1)^{\dim b_i} = \chi(M),$$

as desired! $\square$

The mod 2 proof is similar.