Lecture 4. Stiefel-Whitney Classes.

We now present 4 axioms describing desired properties for a system of cohomology classes on a vector bundle. We don't know that any such classes exist yet.

1. To each vector bundle $\xi$ there is a sequence of cohomology classes

$$\omega_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2)$$

with $i = 0, \ldots, n$. $\omega_0 = 1$ and $\omega_i = 0$ for $i > n$, called the Stiefel-Whitney classes of $\xi$.

2. (Naturality) If $f: B(\xi) \to B(\eta)$ is the base map of a bundle map $\hat{f}: \xi \to \eta$ then

$$\omega_i(\xi) = f^* \omega_i(\eta).$$

3. (Whitney product theorem) If $\xi, \eta$ have the same base space $B$, then
we have

\[ \omega_k(\xi \oplus \eta) = \sum_{i=0}^{k} \omega_i(\xi) \cup \omega_{k-i}(\eta). \]

4. The Stiefel-Whitney class \( \omega_1(\mathbb{C} \mathbb{P}^1) \) of the canonical line bundle over the circle \( \mathbb{C} \mathbb{P}^1 \) is nonzero.

We are now going to work out some consequences of these assumptions without worrying about whether the classes exist.

\[ \text{Proposition 1: If } \xi \cong \eta \text{ then } \omega_i(\xi) = \omega_i(\eta). \]

\text{Proof: Easy consequence of naturality.}

\[ \text{for } i > 0 \]

\[ \text{Proposition 2: If } \xi \text{ is a trivial bundle then } ^0\omega_i(\xi) = 0 \]

\text{Proof: We know that } E(\xi) = \mathbb{R}^n \times B(\xi). \text{ Take the map } f: E(\xi) \to \mathbb{R}^n \times \mathbb{R}^3 \text{ given by } \]

\[ f(b,v) = (p,v) \]
this is continuous, and carries each fiber of $\mathbb{E} \times \mathbb{E}$ isomorphically onto $\mathbb{R}^1$. Thus this is a bundle map onto the trivial bundle over a point, $\varepsilon$! Since a point has no homology in dimensions $> 0$, the $S^1$ classes $\mathbf{w}^i(\varepsilon') = 0$ for $i > 0$. But by naturality, this implies $\mathbf{w}_i(\varepsilon) = 0$ for $i > 0$ since

$$\mathbf{w}_i(\varepsilon) = f^* \mathbf{w}_i(\varepsilon'). \quad \square$$

By the Whitney product theorem,

Proposition 3. If $\varepsilon$ is trivial, $\mathbf{w}_i(\varepsilon \otimes \eta) = \mathbf{w}_i(\eta)$.

This leads directly to a cool consequence:

Proposition 4. If $\varepsilon$ is a Euclidean $\mathbb{R}^n$-plane bundle with $k$ linearly independent cross sections, then

$$\mathbf{w}_n(\varepsilon) = \mathbf{w}_{n-1}(\varepsilon) = \ldots = \mathbf{w}_{n-k+1}(\varepsilon) = 0.$$
Proof. Take the $R^k$ sub-bundle of $\xi$ generated by the cross-sections and call it $\zeta$. We know that $\zeta$ is trivial and that
$$\xi = \zeta \oplus \zeta^\perp$$
<an $n-k$ plane bundle>

But by Prop. 3, this means that
$$\omega_i(\xi) = \omega_i(\zeta^\perp) = \text{the SW classes of an } n-k \text{ plane bundle}$$
so $\omega_i(\xi) = 0$ for $i > n-k$.

Consequence. If $\omega_n(TM) \neq 0$, then $M$ has no nonvanishing vector field. If $\omega_{n-k}(TM) \neq 0$, then $M$ has at most $K$ nowhere dependent vector fields.

Now suppose that $\xi \otimes \eta$ is trivial. We then have a set of relationships between the SW classes of $\xi$ and $\eta$ from the Whitney product formulae.
For instance, we have

\[ \omega_1(\xi \oplus \eta) = 0 = \omega_0(\xi) \cup \omega_1(\eta) + \omega_2(\xi) \cup \omega_0(\eta) \]

Now we have to invoke a fact or two about cohomology and cup products. First, we are implicitly assuming that \( B(-) \) is connected in all our vector bundles. Next, it is a property of cup products that on a connected \( \text{H}^0(B(-); \mathbb{Z}/2) = \mathbb{Z}/2 \) for all our bundles.

Second, it is a property of cup that if \( \alpha \in \text{H}^0(M; R), \beta = R, \beta \in \text{H}^k(M; R) \) then multiplication in \( R \)

\[ \alpha \cup \beta = \alpha \beta \]

We can see this easily in DeRham cohomology, where cup is wedge and 0-classes are 0-forms (functions).

\[ = \omega_1(\eta) + \omega_1(\xi). \text{ (since } \omega_0 = 1 \text{ always).} \]
Similarly (if we eliminate the $U$ for cup when it is understood) we have

$$\omega_2(\xi \ast \eta) = 0 = \omega_2(\xi) + \omega_1(\xi) \omega_1(\eta) + \omega_2(\eta).$$

and so forth. Now if we know $\omega_i(\xi)$, observe that we can solve inductively for $\omega_1(\eta), \omega_2(\eta), \ldots, \omega_n(\eta)$ by following this chain of equalities!

We now formalize this process.

Definition. Let $H^\infty(B; \mathbb{Z}/2)$ be a ring whose elements are formal infinite series in the form

$$a = a_0 + a_2 + a_4 + \ldots,$$

where $a_i \in H^i(B; \mathbb{Z}/2)$.

$ab = (a_0b_0) + (a_0b_2 + a_2b_0) + (a_0b_4 + a_4b_2 + a_2b_0) + \ldots$

(where the operation is $U$ in $H^*(B; \mathbb{Z}/2)$).
and
\[ a + b = (a_0 + b_0) + (a_1 + b_1) + \ldots \]
where the addition is in the groups \( H^1(B; \mathbb{Z}/2) \).

Lemma. The multiplication \( a \cdot b \) is commutative and associative.

Proof. We recall that if \( \alpha \in H^p(B; \mathbb{Z}/2) \), \( \beta \in H^q(B; \mathbb{Z}/2) \), then
\[ \alpha \cdot \beta = \eta^{p \cdot q} \beta \cdot \alpha. \]
However, we are in \( \mathbb{Z}/2 \), so the sign is meaningless and
\[ \alpha \cdot \beta = \beta \cdot \alpha. \]
Thus our product on \( H^{pt} \) inherits commutativity.

Associativity boils down to the observation that the \( k \)-th term of \( abc \) is given by \( \sum_{x+y+z=k} a_x b_y c_z \) in either order.
Definition. The total Stiefel-Whitney class of an $n$-plane bundle $\xi$ over $B$ is defined to be

$$\omega(\xi) = 1 + \omega_1(\xi) + \ldots + \omega_n(\xi) + O + \ldots$$

in $H^*(B; \mathbb{Z}/2)$.

Remark. $\omega(\xi + \eta) = \omega(\xi) \omega(\eta)$ by construction.

Lemma. The collection of series that start with 1 forms a commutative group under multiplication.

Proof. Given $\omega$, we can construct $\overline{\omega}$ (the inverse) inductively assuming $\overline{\omega} \omega = 1 + 0 + 0 + \ldots$. In fact,

$$\overline{\omega_1} = \omega_1$$

$$\overline{\omega_2 + \omega_1 \omega_1 + \omega_2} = 0,$$ so $$\overline{\omega_2} = \omega_1^2 + \omega_2.$$

$$\overline{\omega_3 + \omega_2 \omega_1 + \omega_1 \omega_2 + \omega_3} = 0,$$ so

$$\overline{\omega_3} = (\omega_1^2 + \omega_2) \omega_1 + \omega_1 \omega_2 + \omega_3 = \omega_3 + \omega_4^3.$$
In general, 

\[ \overline{\omega_n} = \overline{\omega_{n-1}} \omega_1 + \overline{\omega_{n-2}} \omega_2 + \ldots + \overline{\omega_1} \omega_{n-1} + \omega_n, \]

so the induction is clear. \( \square \)

Remark. We can write this as a power series

\[
\overline{\omega} = \left[ 1 + (\omega_1 + \omega_2 + \omega_3 + \ldots) \right]^{-1} 
\]

\[
= \left[ 1 - (\omega_1 + \omega_2 + \omega_3 + \ldots) + (\omega_1 + \omega_2 + \omega_3 + \ldots)^2 
- (\omega_1 + \omega_2 + \omega_3 + \ldots)^3 + \ldots \right]^{-1} 
\]

\[
= 1 - \omega_1 + (\omega_1^2 - \omega_2) + (-\omega_1^3 + 2\omega_1\omega_2 - \omega_3) + \ldots 
\]

Eventually, we can use this to conclude

\[
\overline{\omega} = \ldots + \frac{(i_1 + \ldots + i_k)!}{i_1! \ldots i_k!} \omega_1^{i_1} \ldots \omega_k^{i_k} + \ldots 
\]

which is pretty cool.
We now know that if $\xi$ and $\eta$ are bundles over $B$, then we can solve
\[ \omega(\xi \oplus \eta) = \omega(\xi) \omega(\eta) \]
for
\[ \omega(\eta) = \overline{\omega}(\xi) \omega(\xi \oplus \eta). \]
If $\xi \oplus \eta$ is trivial, $\omega(\eta) = \overline{\omega}(\xi)$.

Lemma. (Whitney duality theorem) If $M \subset \mathbb{R}^N$ is a manifold then the Stiefel-Whitney classes of the tangent and normal bundles are related by
\[ \omega_i(V) = \overline{\omega}_i(TM). \]

We can now actually compute some SW classes!
Example. Since the normal bundle $\nu(S^n)$ is trivial for $S^n \subset \mathbb{R}^{n+1}$, it follows that $\omega(\nu) = 1$. Thus $\overline{\omega}(TS^n) = 1$, so $\omega(TS^n) = 1$ as well.

Conclusion. SW classes are not enough to detect the topology of $TS^n$.

We now will consider bundles over $\mathbb{R}P^n$. We first recall

$$H^i(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2 \quad \text{for} \quad 0 \leq i \leq n$$

$$= 0 \quad \text{for} \quad i > n$$

Further, if $a$ is the generator of $H^2(\mathbb{R}P^n; \mathbb{Z}/2)$ then the generator of $H^k(\mathbb{R}P^n; \mathbb{Z}/2)$ is $a^k$. 
Example. The canonical line bundle $y^1_n$ over $\mathbb{RP}^n$ has total SW class $1+a$.

Proof. Include $j: \mathbb{RP}^1 \to \mathbb{RP}^n$. There is a corresponding bundle map (also called $j$) $j: y^1_1 \to y^1_n$, so

$$j^*\omega_1(y^1_n) = \omega_1(y^1_1) \neq 0 \text{ (by axiom)}$$

Thus $\omega_1(y^1_n) \neq 0$. But the only elements in $H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ are 0 and $a$, so $\omega_1(y^1_n) = a$. Since $y^1_n$ is a 1-plane bundle, all higher $\omega_i$ are automatically zero.

Example. The line bundle $y^1_n$ over $\mathbb{RP}^n$ is a subbundle of the trivial $\mathbb{R}^{n+1}$ bundle over $\mathbb{RP}^n$. Let $y^\perp$ be the complement of $y^1_n$. Then

$$\omega(y^\perp) = 1 + a + \ldots + a^n.$$
Proof. $\gamma^n \oplus y^+$ is trivial, so

$$\omega(y^+) = \omega(\gamma^n) = (1 + a)^{-1} = 1 + a + a^2 + \ldots + a^n.$$  

Now this bundle is related to $T\mathbb{RP}^n$, but it is not the same (if confused, recall that the "standard" embedding of $\mathbb{RP}^n$ as a quotient of $S^n$ is not an embedding into $\mathbb{R}^{n+1}$.)

We claim:

Lemma. $T(\mathbb{RP}^n) \cong \text{Hom}(\gamma^n, y^+)$. 

(What does the rhs mean? Recall that last time we claimed that we could construct a new bundle from any functor on vector spaces. Here's an example. Each fiber of

$$F_b(\text{Hom}(\gamma^n, y^+)) = \text{Hom}(F_b(\gamma^n), F_b(y^+)).$$

\begin{tikzcd}
\mathbb{R}^1 & \mathbb{R}^n \\
\mathbb{R}^1 \\
\end{tikzcd}
Proof. Let $L$ be a line through the origin (a point in $\mathbb{RP}^n$) in $\mathbb{R}^{n+1}$, intersecting $S^n$ in $\pm x$. Let $L^\perp$ be the complementary $n$-plane. Let $f : S^n \to \mathbb{RP}^n$ be the canonical double cover map. Consider

$$Df : T S^n \to T(\mathbb{RP}^n).$$

At $\pm x$, $Df$ maps

$$T_x S^n \to T_{\pm x} \mathbb{RP}^n$$
$$T_{-x} S^n \to T_{\pm x} \mathbb{RP}^n$$

So we claim $Df(x,v) = Df(-x,-v)$, and $T S^n$ double covers $T(\mathbb{RP}^n)$. Now we can then write

$$T(\mathbb{RP}^n) = \big\{ (x,v), (-x,v) \in T S^n \times T S^n \big\}$$

$$= \big\{ (x,v), (-x,v) \in (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})^* \big| x \cdot x = 1, x \cdot v = 0 \big\}.$$
Given such a pair, we can define a linear map \( l : L \to L^+ \) by \( l(x) = v \).
(And given such a linear map, we can define \( v \).)

So \( T_{\mathbb{R} \times x^3} \mathbb{R}P^n \) is canonically isomorphic to \( \text{Hom}(L, L^+) \). Since the iso. is canonical, this implies it is cts. in \( L, L^+ \) and \( T(\mathbb{R}P^n) \cong \text{Hom}(y_1, y^+) \). \( \square \)

It would be great if this let us compute \( \omega(T(\mathbb{R}P^n)) \). But we can't quite do this yet, directly from our description of \( T(\mathbb{R}P^n) \). However, we have

**Theorem.** If \( E_4 = \mathbb{R} \times \mathbb{R}P^n \), then

\[
T(\mathbb{R}P^n) \oplus E_4 = y_1 \oplus y_2 \oplus \ldots \oplus y_n.
\]
The proof of this theorem is absolutely cute!

Proof. The bundle \( \text{Hom}(\gamma_n^1, \gamma_n^4) \) is a 1-plane bundle, since the fiber is \( \text{Hom}(\mathbb{R}, \mathbb{R}) \) which is \( \mathbb{R}^* = \mathbb{R}^4 \). Further it has a nonvanishing cross-section so it is the trivial 1-plane bundle \( \mathcal{E}_4 \).

So

\[
\text{T}(\mathbb{R}P^n) \oplus \mathcal{E}_4 = \text{Hom}(\gamma_n^1, \gamma^1) \oplus \text{Hom}(\gamma_n^4, \gamma_n^4) \\
= \text{Hom}(\gamma_n^1, \gamma^1 \oplus \gamma_n^4) \\
= \text{Hom}(\gamma_n^1, \mathcal{E}^{n+1}) \\
= \text{Hom}(\gamma_n^1, \mathcal{E}_4 \oplus \cdots \oplus \mathcal{E}_4) \\
= \text{Hom}(\gamma_n^4, \mathcal{E}_4) \oplus \cdots \oplus \text{Hom}(\gamma_n^4, \mathcal{E}_4).
\]

But \( \gamma_n^4 \) is Euclidean (since it is a 1-plane bundle), so fiber by fiber, \( \text{Hom}(F_b(\gamma_n^4), F_b(\mathcal{E}_4)) \)

\[
= \text{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R}^* \rightarrow = F_b(\gamma_n^4).
\]

canonically
Corollary.

\[
\omega(T(\mathbb{RP}^n)) = \omega(T(\mathbb{RP}^n) \oplus \varepsilon^1)
= \omega(\gamma_n^4 \oplus \ldots \oplus \gamma_n^2)
= \omega(\gamma_n^4) \cdots \omega(\gamma_n^2)
= (1+a) \cdots (1+a)
= (1+a)^{n+1}
= 1 + \binom{n+1}{1}a + \binom{n+1}{2}a^2 + \ldots + \binom{n+1}{n}a^n
\]

(Remember that \(a^{n+1} = 0\) in \(H^*(\mathbb{RP}^n)\).)

Examples.

\[
\begin{array}{c|c|c|c|c|c|c|c}
\omega(\mathbb{RP}^2) &=& 1 + a + a^2 \\
\omega(\mathbb{RP}^3) &=& 1 \\
\omega(\mathbb{RP}^4) &=& 1 + a + a^4 \\
\omega(\mathbb{RP}^5) &=& 1 + a + a^4 + a^5 \\
\end{array}
\]
Corollary. The only projective spaces that can be parallelizable are $\mathbb{RP}^{2^n-1}$, since these are the only $\mathbb{RP}^k$ with $\omega(\mathbb{RP}^k) = 1$.

Proof. Recall $(a+b)^2 = a^2 + b^2$ mod 2. So since we are in $\mathbb{Z}/2\mathbb{Z}$ coefficients, we have

$$(1 + a)^{2^n} = 1 + a^{2^n}$$

So if $2^n = n+1$, then

$$\omega(T(\mathbb{RP}^n)) = (1 + a)^{n+1} = (1 + a)^{2^n} = 1 + a^{2^n} = 1 + a^{n+1} = 1.$$ 

Now if $n+1$ is not a power of 2, then $n+1 = 2^m \cdot m$ with $m \geq 1$ odd. We observe

$$\omega(T(\mathbb{RP}^n)) = (1 + a)^{n+1}$$

$$= (1 + a^{2^m})^m$$

$$= 1 + m \cdot a^{2^m} + \binom{m}{2}(a^{2^m})^2 + \ldots.$$ 

But $2^m < n+1$, so $a^{2^m} \neq 0$, and $m$ odd, so $ma^{2^m} \neq 0$ and $\omega(T(\mathbb{RP}^n)) \neq 1$. □
Remark. We will prove $\mathbb{RP}^1, \mathbb{RP}^3, \mathbb{RP}^7$ are parallelizable in a minute. But it is known that $\mathbb{RP}^{15}, \mathbb{RP}^{31}, \ldots$ are not parallelizable for more subtle reasons.

Theorem. Suppose $E$ a bilinear product

$$p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

without zero divisors. Then $\mathbb{RP}^{n-1}$ is parallelizable.

Proof. Let $b_1, \ldots, b_n$ be the standard basis for $\mathbb{R}^n$. There is a map

$$A : y \mapsto p(y, b_1)$$

from $\mathbb{R}^n \to \mathbb{R}^n$. Since $p$ has no zero divisors, this map is linear and has no kernel (other than $y \neq 0$) so it is an isomorphism. Now we define $n$ maps $v_i : \mathbb{R}^n \to \mathbb{R}^n$ by

$$v_i( p(y, b_1) ) = p(y, b_i)$$

or

$$v_i(x) = p(A^{-1}(x), b_i).$$
We claim that
\[ V_1(x), \ldots, V_n(x) \text{ are lin. indep. for } x \neq 0 \]
and \( V_1(x) = x \).

Suppose
\[ a_1 V_1(x) + \ldots + a_n V_n(x) = 0, \text{ all } a_i \neq 0 \]

Well,
\[ a_1 V_1(x) + \ldots + a_n V_n(x) = a_1 p(A^{-1}(x), b_1) + \ldots + a_n p(A^{-1}(x), b_n) \]
\[ = p(A^{-1}(x), a_1 b_1 + \ldots + a_n b_n), \]
so we have found zero divisors \( A^{-1}(x) \) (\( \neq 0 \) since \( A \) is an isomorphism and \( x \neq 0 \)) and \( \sum a_i b_i \) (\( \neq 0 \) since not all \( a_i \) are zero).

Further,
\[ V_1(x) = p(A^{-1}(x), b_1) = A(A^{-1}(x)) = x. \]
Claim: $v_2, \ldots, v_n$ give rise to $n-1$ nowhere dependent cross sections of

$$T(\mathbb{RP}^{n-1}) \cong \text{Hom} (\mathbb{R}^{n-1}, \mathbb{R}^1).$$

Let $L$ be some line through the origin. We define a map

$$\overline{v_i} : L \to L^\perp$$

by for $x \in L$, let $\overline{v_i}(x) = \pi(v_i(x))$ where $\pi : \mathbb{R}^n \to L^\perp$ is orthogonal projection down $L$.

1) $\overline{v_1} = 0$. Since $v_1(x) = x$ is on $L$.

2) $\overline{v_2}, \ldots, \overline{v_n}$ are lin. indep.

   If not, then

   \[ a_2 \overline{v_2} + \ldots + a_n \overline{v_n} = 0 \]

   or

   \[ a_2 (v_2 + b_2 x) + \ldots + a_n (v_n + b_n x) = 0 \]

   \[ (a_2 b_2 v_2 + \ldots + a_n b_n v_n) + a_2 v_2 + \ldots + a_n v_n = 0 \]
and \( v_1, \ldots, v_n \) are linearly dependent. 

Thus \( \overline{v}_2, \ldots, \overline{v}_n \) trivialize \( \text{Hom}(y_{n-1}, y^1) \) and hence \( T(1R\mathbb{P}^{n-1}) \) is trivial. \( \square \)

Examples.

\( 1R^2 = \) complex numbers

\( 1R^4 = \) quaternions

\( p \rightarrow i \)

\( k \leftarrow j \)

Quaternions product of pure quaternions

\( q_a = q_1 i + q_2 j + q_3 k \)

\( p = p_1 i + p_2 j + p_3 k \)

can be written in terms of \( 1R^3 \) vectors

\( \vec{q} = (q_1, q_2, q_3), \ \vec{p} = (p_1, p_2, p_3) \)

\( qp = \vec{q} \cdot \vec{p} + (\vec{q} \times \vec{p})_1 i + (\vec{q} \times \vec{p})_2 j + (\vec{q} \times \vec{p})_3 k. \)

\( 1R^7 = \) octonions & or Cayley numbers.

(n.b. Look this up!)
Immersions.

Which $\mathbb{R}P^n$ can be embedded in $\mathbb{R}^{n+k}$?

By Whitney duality, if $M^n$ immerses in $\mathbb{R}^{n+k}$ then

$$\omega_i(v) = \overline{\omega}_i(M)$$

implies $\overline{\omega}_i(M) = 0$ for $i > k$.

Example. Consider $\mathbb{R}P^9$.

$$\omega(\mathbb{R}P^9) = (1 + a^8)^{10}$$

$$= 1 + a^2 + a^8.$$ 

$\overline{\omega}(\mathbb{R}P^9)$ is obtained by solving

$$\overline{\omega}_0 = 1$$
$$\overline{\omega}_1 = \omega_1 = 0.$$
$$\overline{\omega}_2 = \overline{\omega}_1 \omega_1 + \omega_2 = 1.$$ 
$$\overline{\omega}_3 = \overline{\omega}_2 \omega_1 + \overline{\omega}_2 \omega_2 + \omega_3 = 0$$
$$\overline{\omega}_4 = \overline{\omega}_3 \omega_1 + \overline{\omega}_3 \omega_2 + \overline{\omega}_3 \omega_3 + \omega_4 = 1.$$
\[ \overline{\omega}_5 = \overline{\omega}_4 \overline{\omega}_1 + \overline{\omega}_3 \overline{\omega}_2 + \overline{\omega}_2 \overline{\omega}_4 + \overline{\omega}_4 \overline{\omega}_5 + \frac{1}{\overline{\omega}_5} \]

= 0

\[ \overline{\omega}_6 = \overline{\omega}_5 \overline{\omega}_1 + \overline{\omega}_4 \overline{\omega}_2 + \overline{\omega}_3 \overline{\omega}_3 + \overline{\omega}_2 \overline{\omega}_4 + \overline{\omega}_4 \overline{\omega}_5 + \frac{1}{\overline{\omega}_6} \overline{\omega}_6 \]

= 1

\[ \overline{\omega}_7 = \overline{\omega}_6 \overline{\omega}_1 + \overline{\omega}_5 \overline{\omega}_2 + \overline{\omega}_4 \overline{\omega}_3 + \overline{\omega}_3 \overline{\omega}_4 + \overline{\omega}_2 \overline{\omega}_5 + \overline{\omega}_4 \overline{\omega}_6 + \frac{1}{\overline{\omega}_7} \overline{\omega}_7 = 0. \]

\[ \overline{\omega}_8 = \overline{\omega}_7 \overline{\omega}_1 + \overline{\omega}_6 \overline{\omega}_2 + \overline{\omega}_5 \overline{\omega}_3 + \overline{\omega}_4 \overline{\omega}_4 + \overline{\omega}_3 \overline{\omega}_5 + \overline{\omega}_2 \overline{\omega}_6 + \overline{\omega}_4 \overline{\omega}_7 + \frac{1}{\overline{\omega}_8} \overline{\omega}_8 = 0. \]

So

\[ \overline{\omega}(\text{IRP}^9) = 1 + a^2 + a^4 + a^6. \]

This means that if \( \text{IRP}^9 \) immerses in \( \text{IR}^{9+k} \), then \( k \geq 6 \). Cool!

The strongest results along these lines come from \( \text{IRP}^{2n} \).
If \( n = 2^r \),
\[
\omega(RP^n) = \omega(RP^{2^r}) \\
= (1 + a)^{2^r+1} \\
= (1 + a)^{2^r} (1 + a) \\
= (1 + a^{2^r}) (1 + a) \\
= 1 + a + a^n
\]

Then we can compute
\[
\overline{\omega}_{i-1}(RP^n) = \sum_{k=1}^{0} \sum_{i-k}^{n} \overline{\omega}_{i-k}
\]
\[
= \overline{\omega}_{i-1} \bigg\{ \bigcup_{i \in 1, \ldots, n-1} \bigg\} \\
= \overline{\omega}_{i-1} + \omega_{n} \bigg\{ \bigcup_{i \in n} \bigg\}
\]
\[
= 1 \bigg\{ \bigcup_{i \in 1, \ldots, n-1} \bigg\} 0 \bigg\{ \bigcup_{i = n} \bigg\}
\]

so
\[
\overline{\omega}_{0}(RP^n) = 1 + a + a^2 + \ldots + a^{n-1}
\]
Theorem. If $\mathbb{RP}^{(2^r)}$ can be immersed in $\mathbb{R}^{2^r+k}$, then $k \leq 2^r - 1$, or $\mathbb{RP}^{(2^r)}$ cannot be immersed in any $\mathbb{R}^n$ smaller than $\mathbb{R}^{2^r - 1}$.

Proof. We just gave it.

Remark. Whitney Immersion Theorem shows that any $M^n$ immerses in $\mathbb{R}^{2n-1}$, so this example shows that this theorem is optimal.

Stiefel-Whitney Numbers.

At the moment, the Stiefel-Whitney classes depend on the cohomology ring of the manifold, so there is no way to directly compare the classes of different manifolds. Let's fix that!
Let $M$ be a closed, smooth $n$-manifold, and let

$$\nu_M \in H_n(M; \mathbb{Z}/2)$$

be the fundamental class of $M$, (or top class).

Suppose

$$r_1 + 2r_2 + \ldots + nr_n = \eta$$

then given any vector bundle $E$ over $M$, we can form

$$\omega_1(E)^{r_1} \cdots \omega_n(E)^{r_n} \in H^n(M; \mathbb{Z}/2).$$

**Definition.** $\langle \omega_1(TM)^{r_1} \cdots \omega_n(TM)^{r_n}, \nu_M \rangle$

or $\omega_1^{r_1} \cdots \omega_n^{r_n}[M]$ is the Stiefel-Whitney number associated with $\omega_1^{r_1} \cdots \omega_n^{r_n}$.

Two $n$-manifolds have the same SW numbers if these agree for all such monomials with $\sum r_k = \eta$. 
Example. SW numbers of $\text{IRP}^n$.

We know $\omega(\text{IRP}^n) = (1 + a)^{n+1}$

$$= 1 + (n+1)a + \ldots + (n+1)a^n$$

where the intermediate guys are binomial coefficients in $\mathbb{Z}/2$.

If $n$ is even,

$$\omega_n[\text{IRP}^n] = (n+1)a^n \neq 0,$$

$$\omega_1[\text{IRP}^n] = (n+1)a^n \neq 0.$$

(If $n$ is a power of 2, $\omega(\text{IRP}^n) = 1 + a + a^n$, and this is it for the SW numbers.)

If $n$ is odd, let $n = 2k - 1$, so

$$\omega(\text{IRP}^n) = (1 + a)^{2k} = (1 + a^2)^k,$$

and $\omega_j(\text{IRP}^n) = 0$ when $j$ is odd.

Now suppose

$$r_1 + 2r_2 + \ldots + n\cdot n = n = \text{odd}.$$
Then one of the terms, say $j r_j$ is odd, so $j$ and $r_j$ are both odd.

But $w_j (\mathbb{RP}^n) = 0$, so $w_j (\mathbb{RP}^n) = 0$ and thus the entire monomial

$$w_1^{r_1} \cdots w_n^{r_n} (\mathbb{RP}^n) = 0.$$ 

So all SW numbers vanish for $\mathbb{RP}^{\text{odd}}$.

You would think from this that SW numbers are pretty weak. However, the following theorems show that this is not the case.

Theorem. If $M^n$ is the boundary of a smooth compact [Pontrjagin] compact $n$-1 manifold $B$, then the SW numbers of $M$ are all zero.
Proof. Let $\nu_B \in H_{n+1}(B, M)$ be the top class of $B$. We know

$\partial : H_{n+1}(B, M) \to H_n(M)$

takes $\nu_B$ to $\nu_M$. Further, for any $v \in H^n(M)$,

$\langle v, \partial \nu_B \rangle = \langle \delta v, \nu_B \rangle$

(where $\delta : H^n(M) \to H^{n+1}(B, M)$ is the map induced by $\partial$ under the Hom functor)

that takes homology to cohomology.

Now

$TB$ restricted to $M$ has $TM$ as a sub-bundle.

If we let $TB$ be a Euclidean bundle, there is a unique outward direction on $M = \partial B$, so

$TB|_M \cong TM \oplus E^1$

so the SW classes of $TB|_M$ are equal to the SW classes of $TM$. 
Now the inclusion \( M \hookrightarrow B \) gives us the (cohomology) exact sequence of the pair \((B, M)\)

\[
H^n(B) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(B, M)
\]

Now we observe that by naturality, since the inclusion \( M \hookrightarrow B \) is covered by the bundle map from \( TB|_M \rightarrow TB \), we know that the SW classes of \( TB|_M \) are the restrictions of the SW classes of \( TB \) to \( M \). These live in \( H^n(B) \).

In particular, if \( \omega_1^M \cdots \omega_n^M \in H^n(M) \) is a product of SW classes of \( TM \), it is the image under \( i^* \) of corresponding classes \( \omega_1^B \cdots \omega_n^B \in H^n(B) \).

Thus \( \delta(\omega_1^B \cdots \omega_n^B) = 0 \) for all such monomials.
And for any SW number of $M$,
\[
\langle \omega_1^a \cdots \omega_n^a, \nu_M \rangle = \langle \omega_1^a \cdots \omega_n^a, \partial \nu_B \rangle = \langle \delta(\omega_1^a \cdots \omega_n^a), \nu_B \rangle = 0. \quad \Box
\]

**Theorem. [Thom]** If all SW numbers of $M$ are zero, then $M = \partial B$ for some smooth compact $B$.

**Proof.** Beyond scope of this class. (Hard.)

**Example.** $\mathbb{RP}^{odd}$ has all SW numbers 0, so $\mathbb{RP}^{odd}$ is a boundary.

$\mathbb{RP}^1 = S^1 = \partial D^2$

$\mathbb{RP}^3 = \partial \text{(what?)}$
Definition. Smooth closed manifolds $M_1$ and $M_2$ are (unoriented) cobordant if $M_1 \cup M_2$ is the boundary of a smooth compact $(n+1)$ manifold.

Corollary. If $M_1, M_2$ are smooth closed $n$-manifolds, then $M_1$ and $M_2$ are cobordant if and only if all SW numbers are equal.

Proof. All SW numbers of $M_1 \cup M_2$ are even, hence zero. Now apply Thom.

Problems. 4A-4C.