Lecture 2. Vector Bundles

A bundle is best understood as a kind of "twisted product" of two manifolds.

\[ F = \text{fiber} \quad \rightarrow \quad E = \text{total space} \]

\[ S^4 \times S^4 = T^2 \]

\[ \Pi = \text{projection map} \]

\[ B = \text{base space} \]

A product space is one example, but a trivial one. Formally, we have

Definition. A real vector bundle \( \xi \) over \( B \) consists of

1) a topological space \( E \) called the total space
2) a (continuous) map $\pi: E \to B$ called the projection map

3) for each $b \in B$ a vector space structure on $\pi^{-1}(b)$.

which obey the following conditions:

For each $b \in B$, $\exists$ a neighborhood $U$, an integer $n$, and a homeomorphism

$$h: U \times \mathbb{R}^n \to \pi^{-1}(U)$$

so that for each $b \in U$, $x \mapsto h(b, x)$ is an isomorphism $\mathbb{R}^n \to \pi^{-1}(b)$.

We call $(U, h)$ a local coordinate system for $\xi$ about $b$.

If $U = B$, the bundle is trivial.

We call $\pi^{-1}(b)$ the fiber over $b$. 
We usually assume that all fibers have the same dimension, \( n \), making \( \xi \) an \( n \)-plane bundle.

If everything is smooth and \( \varphi \) is a \textit{diffeomorphism}, then this is a \textit{smooth vector bundle}.

**Definition.** Two vector bundles over the same space are \textit{isomorphic} iff \( \exists \) a homeomorphism \( f : E(\xi_1) \rightarrow E(\xi_2) \) so that \( F_b(\xi_1) \xrightarrow{f} F_b(\xi_2) \) is an isomorphism for all \( b \).

**Example.** \( B \times \mathbb{R}^n \) is the trivial bundle over \( B \) if we let \( \pi(b, x) = b \) and take the vector space structure

\[
\pi^{-1}(b) = \{ (b, x) : x \in \mathbb{R}^n \}
\]

on \( \pi^{-1}(b) \).
Example. The tangent bundle, $TM$.

If the tangent bundle is trivial, we say $M$ is parallelizable.

Example. $T^2$ is parallelizable (prove it!)

$S^2$ is not parallelizable (by $x 
eq 0$)

so we have our first example of a nontrivial vector bundle in $T S^2$.

Example. If $M \subset \mathbb{R}^n$, we can define the normal bundle of $M$ to be the subspace of $M \times \mathbb{R}^n$ of pairs $(x, v)$ s.t. $v \perp T_x M$

This is usually denoted $v$. 
Question. Is the normal bundle of $S^2$ trivial?

Question. Suppose you can embed $M$ in $\mathbb{R}^{n+1}$. Does that mean that $\nu$ is trivial? Immerse?

Example. Define $\mathbb{RP}^n$ as usual. The canonical line bundle over $\mathbb{RP}^n$ is the $\mathbb{R}$-bundle given by the subset of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ of $(x, \nu)$ so that $\nu$ is a multiple of $x$.

We call this bundle $\gamma^1_n$.

It is clear that $\gamma^1_n$ is locally trivial.

Theorem. If $n > 1$, $\gamma^1_n$ is not trivial.

We will prove this using cross-sections.

Definition. A cross section $s$ of $E$ is a continuous function $s: B \to E$ which
maps \( b \mapsto F_b(\xi) \). The cross-section is nowhere zero if \( s(b) \neq 0 \) for all \( b \).

Note that a vector field is a cross-section of the tangent bundle.

Proof. The trivial bundle has a nowhere zero cross-section. So suppose \( s: \mathbb{R}P^n \to E(\gamma^n) \) is a cross-section. Consider

\[
f: S^n \to \mathbb{R}P^n \xrightarrow{s} E(\gamma^n).
\]

Now \( f(x) = (\xi \pm x, t(x)x) \) where \( t(x) \) is a continuous function of \( x \). And

\[
f(-x) = f(x), \text{ so } t(-x) = -t(x).
\]

Now \( S^n \) is connected, so \( t(x_0) = 0 \) for some \( x_0 \) on any path from \( x \) to \(-x\).
Explicit example. $\mathcal{Y}_4$.

We identify $\mathbb{R}P^2$ with the upper half-circle (with endpoints identified).

Note that over this interval we have an (open) strip, but the gluing reverses orientation on the boundary line.

Hence $\mathcal{Y}_4 = \text{open mobius strip}$, while the trivial $S^4 \times \mathbb{R}^4$ bundle is the cylinder.
Now suppose we have \( \xi \) a bunch of cross-sections \( \xi S_1, \ldots, S_n \xi \) of \( \xi \).

**Definition.** A collection \( \xi S_1, \ldots, S_n \xi \) of cross sections of \( \xi \) are **nowhere-dependent** if for each \( b \in B \), \( S_1(b), \ldots, S_n(b) \) are linearly independent.

**Lemma.** Let \( \xi, \eta \) be vector bundles over \( B \). If \( f: E(\xi) \to E(\eta) \) is continuous and maps each \( F_b(\xi) \) isomorphically onto \( F_b(\eta) \) then \( f \) is a homeomorphism and \( \xi \cong \eta \).

**Proof.** By assumption, \( \begin{array}{ccc} E & \xrightarrow{f} & E \\ \downarrow \pi & & \downarrow \pi \\ B & \xrightarrow{\text{commutes}} & B \end{array} \)

and \( f \) is 1-1, onto and continuous. It remains only to show \( f^{-1} \) is continuous, which follows from the fact that matrix inverses are cts.
Theorem. An \( \mathbb{R}^n \)-bundle \( \xi \) is trivial \( \iff \) it admits \( n \) cross-sections which are nowhere dependent.

Proof. Easy consequence of Lemma.

Examples.

\( S^1 \) and \( S^3 \) are parallelizable.

Euclidean vector bundles.

We can put a little more structure on our vector bundles by asking for an inner product (determined by a pos.def. quadratic form) on each vector space.

Definition. A Euclidean vector bundle is a real vector bundle together with a continuous function \( \mu : E(\xi) \to \mathbb{R} \).
which is quadratic on each fiber.

This is called a\textit{ Euclidean metric} on $\xi$, and if $\xi = \text{TM}$, a\textit{ Riemannian metric} on $M$.

Note that $\mathbb{R}^n$ has the standard metric and the inclusion $\text{M} \hookrightarrow \mathbb{R}^n$ induces

\[ \text{TM} \hookrightarrow \text{TR}^n \]

which makes any $\text{M} \subset \mathbb{R}^n$ Riemannian.

\underline{Lemma}. If $\xi$ is a trivial $n$-plane bundle and $\nu$ is a Euclidean metric on $\xi$, then there exist orthonormal cross-sections $s_1, \ldots, s_n$.

\underline{Proof}. Continuity of \textit{Gram-Schmidt} orthogonalization.

\underline{Problems}. 2A-2B on p.23