Clairaut's Relation

Proposition. The geodesics on a surface of revolution satisfy $r\cos\varphi = \text{constant}$ where r is the distance to the axis and φ the angle with a parallel.

Any constant speed curve with $r\cos\varphi = \text{const.}$ which is not a parallel is a geodesic.

Proof. We have $E=1$, $F=0$, $G=f(u)^2$, assuming

$$x(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $f'(u)^2 + g'(u)^2 = 1$. It's a cool homework exercise to show all Christoffel symbols are 0 except

$$\Gamma^v_{uv} = \frac{f'(u)}{f(u)}, \quad \Gamma^u_{vv} = -f(u)f'(u).$$

so our geodesic equations become

$$u'' - f(u)\,f'(u)\,(v')^2 = 0$$

$$v'' + \frac{2f'(u)\,u'}{f}\,v' = 0$$

Again, the second looks appealing. We can rearrange it as

$$\frac{v''}{v'} = -\frac{2f'(u)\,u'(t)}{f(u(t))}$$

Integrate w.r.t. t to get

$$\ln v'(t) = -2\ln f(u(t)) + C$$

$$v'(t) = \frac{c}{f(u(t))^2},$$

so (multiplying by $f(u(t))^2$),

$$f^2(u(t))\,v'(t) = c$$

Now recall, $G = f^2(u)$. So since our geodesic has tangent vector

$$\alpha'(t) = u'(t)\vec{X}_u + v'(t)\vec{X}_v$$

we have

$$\langle \alpha'(t), \vec{X}_v \rangle = v'(t)\langle \vec{X}_v, \vec{X}_v \rangle$$

$$= v'(t)\, G = \text{constant}.$$

so

$$|\vec{\alpha}'(t)|\,|\vec{X}_v| \cos\varphi = \text{constant}.$$

But $|\alpha'(t)|$ is constant anyway along a geodesic, and $|\vec{X}_v| = f(u) = r$, so $r\cos\varphi = \text{constant}$.

Conclusion: geodesic $\Rightarrow$ $r\cos\varphi = \text{constant}$.

The almost-converse claim is

$$|\vec{\alpha}'(t)| = \text{constant and } r\cos\varphi = \text{constant}$$

$$\Rightarrow \alpha(t) \text{ a parallel or } \alpha(t) \text{ a geodesic}$$

So let's prove that!

$$|\alpha'(t)|^2 = (u')^2 + G(v')^2 = (u')^2 + (f(u))^2(v')^2$$

so if this is constant, ← in t!

$u'(t)$ ← chain rule.

$$0 = 2u'u'' + 2f(u)f'(u)\overset{u'(t)}{\wedge}(v')^2 + 2f(u)^2v'v''$$

If $r\cos\varphi =$ constant, we know (doing the previous argument backwards)

$$v''(t) = -\frac{2f'u'v'}{f} \Big\} \text{the second geodesic equation.}$$

so (substituting above)

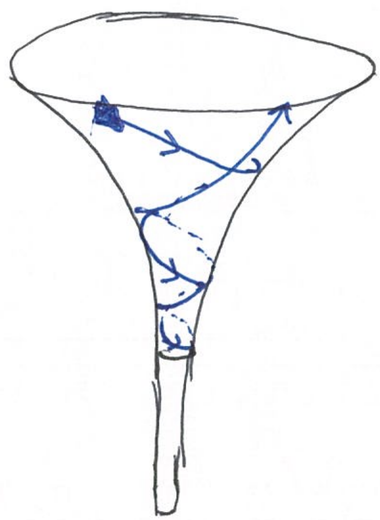
$$0 = 2u'u'' + 2f(u)\overset{u'(t)}{\wedge}f'(u)(v')^2 - \frac{\overset{4}{\cancel{2}}f^{\cancel{2}}(v')^2f'u'}{\cancel{f}}$$

$$= u'(u'' + \cancel{2}ff'(v')^2 - \overset{2}{\cancel{4}}ff'(v')^2)$$

$$= u'(\underbrace{u'' - ff'(v')^2}_{\text{the first geodesic equation!}})$$

Thus unless $u'(t) \equiv 0$, our curve satisfies both geodesic equations. $\square$

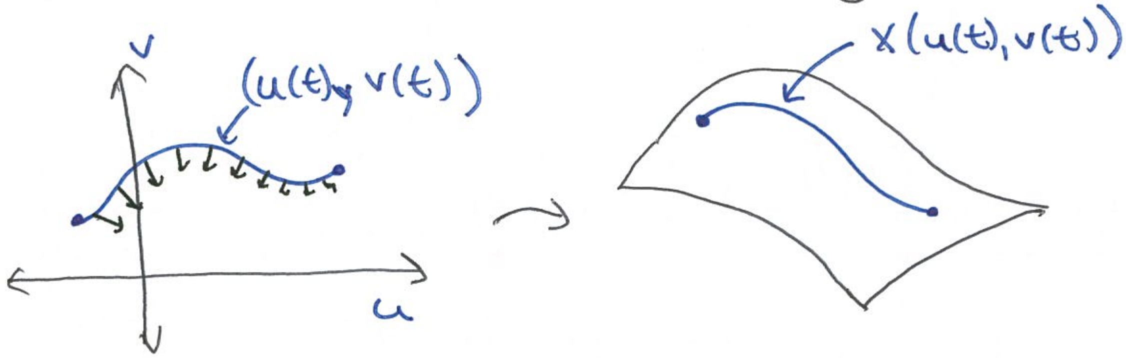
Corollary. A geodesic on a surface of revolution asympotic to the axis



reaches $z = -\infty$ $\Leftrightarrow$ it is a ~~me~~ meridian.

Proof. $r\cos\varphi \leq r$, so if $r \to 0$ as $z \to -\infty$, we eventually reach some $r =$ the constant for the geodesic. At this point $\cos\varphi = 1$, so α is (briefly) tangen to the parallel!

We now show that a geodesic has (locally) shortest length:



As before, we take a parametrized curve $\alpha(t) = (u(t), v(t))$ and consider the variational vector fields

$$\frac{d}{dx} \alpha_x(t) \Big|_{x=0} = W(t), \quad W(0) = W(1) = \vec{0}.$$

But now the functional is

$$\text{Length} = \int_0^1 \sqrt{E(u(t), v(t))\, u'(t)^2 + 2F u'v' + G(v')^2}\, dt$$

Taking the x derivative and assuming $\alpha(t)$ was parametrized so this $= 1$, we have to be careful about the chain rule

$$\frac{d}{dx}\Big(E(u_x(t),v_x(t))\left(\frac{\partial}{\partial t}u_x(t)\right)^2 + 2F(u_x(t),v_x(t))\frac{\partial}{\partial t}u_x(t)\frac{\partial}{\partial t}v_x(t) + G(u_x(t),v_x(t))\left(\frac{\partial}{\partial t}v_x\right)^2\Big)^{1/2} =$$

$$\frac{1}{2}(\text{the whole thing})^{-1/2}\Big(\left\langle \begin{bmatrix}E_u\\E_v\end{bmatrix}, \begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial x}\end{bmatrix}\right\rangle \left(\frac{\partial}{\partial t}u_x\right)^2$$

$$+ E\cdot 2\left(\frac{\partial}{\partial t}u\right)\left(\frac{\partial^2}{\partial x\partial t}u\right) + 2\left\langle \begin{bmatrix}F_u\\F_v\end{bmatrix}, \begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial x}\end{bmatrix}\right\rangle \frac{\partial}{\partial t}u\frac{\partial}{\partial t}v$$

$$+2F\left(\frac{\partial^2}{\partial x\partial t}u\frac{\partial}{\partial t}v + \frac{\partial}{\partial t}u\frac{\partial^2}{\partial x\partial t}v\right) + \left\langle \begin{bmatrix}G_u\\G_v\end{bmatrix}, \begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial x}\end{bmatrix}\right\rangle \left(\frac{\partial}{\partial t}v\right)^2$$

$$+ \cancel{2}G\cdot 2\left(\frac{\partial}{\partial t}v\right)\left(\frac{\partial^2}{\partial x\partial t}v\right)\Big)$$

Now the initial curve may as well ~~a~~ have been arclength parametrized, so (the whole thing) = 1. Further,

$$\left(\frac{\partial}{\partial x}u, \frac{\partial}{\partial x}v\right) = \vec{W} = (W_u, W_v)$$

so we get

$$\frac{1}{2}\left\langle u'(t)^2 \begin{bmatrix} E_u \\ E_v \end{bmatrix}, \vec{W} \right\rangle$$

$$+ E\, u'(t)\, W_u' + \left\langle u'v' \begin{bmatrix} F_u \\ F_v \end{bmatrix}, \vec{W} \right\rangle$$

$$+ F\left(v' W_u' + u' W_v'\right) +$$