Clairaut's Relation

Proposition. The geodesics on a surface of revolution satisfy \( r \cos \phi = \text{constant} \) where \( r \) is the distance to the axis and \( \phi \) the angle with a parallel. Any curve with \( r \cos \phi = \text{const.} \), which is not a parallel is a geodesic.

Proof. We have \( E=1, F=0, G=f(u)^2 \), assuming

\[
x(u,v) = (f(u) \cos v, f(u) \sin v, g(u))
\]

where \( f''(u)^2 + g'(u)^2 = 1 \). It's a cool homework exercise to show all Christoffel symbols are 0 except

\[
\Gamma^v_{uw} = \frac{f'(u)}{f(u)}, \quad \Gamma^u_{vw} = -f(u)f''(u).
\]
so our geodesic equations become

\[ u'' - f(u) f'(u) (v')^2 = 0 \]
\[ v'' + 2f'(u) u' v' = 0 \]

Again, the second looks appealing. We can rearrange it as

\[ \frac{v''}{v'} = - \frac{2f'(u) u'(t)}{f(u(t))} \]

Integrate w.r.t. \( t \) to get

\[ \ln v'(t) = -2 \ln f(u(t)) + C \]

\[ v'(t) = \frac{C}{f(u(t))^2} \]

so (multiplying by \( f(u(t))^2 \)),

\[ f^2(u(t)) v'(t) = C \]
Now recall, \( G = f^2(u) \). So since our geodesic has tangent vector
\[
\alpha'(t) = u'(t) \hat{x}_u + v'(t) \hat{x}_v
\]
we have
\[
\langle \alpha'(t), \hat{x}_v \rangle = v'(t) \langle \hat{x}_v, \hat{x}_v \rangle
\]
\[
= v'(t) G = \text{constant}.
\]
so
\[
|\alpha'(t)| |\hat{x}_v| \cos \Phi = \text{constant}.
\]
But \( |\alpha'(t)| \) is constant anyway along a geodesic, and \( |\hat{x}_v| = f(u) = r, \) so \( r \cos \Phi = \text{constant} \).

Conclusion: geodesic \( \Rightarrow r \cos \Phi = \text{constant} \).

The almost-converse claim is
\[
|\alpha'(t)| = \text{constant and } r \cos \Phi = \text{constant}
\]
\[\Rightarrow \alpha(t) \text{ a parallel or } \alpha(t) \text{ a geodesic} \]
So let's prove that!

\[ |x'(t)|^2 = (u')^2 + g - (v')^2 = (u')^2 + (f'(u))^2(v')^2 \]

so if this is constant, \[ u'(t) \text{ chain rule.} \]

0 = 2u'u'' + 2f(u)f'(u)(v')^2 + 2f'(u)^2v'v''

If \( r \cos \theta \) is constant, we know (doing the previous argument backwards)

\[ v''(t) = -\frac{2f'u'v'}{f} \] \( \{ \) the second geodesic equation.

so (substituting above)

\[ u'(t) = \frac{4}{f} \]

0 = 2u'u'' + 2f(u)f'(u)(v')^2 - \frac{2f'(u)^2f'u'}{f}

= \[ u'(u'' + \frac{2f'(u')^2}{2} - \frac{2}{2f'}(v')^2) \]

= \[ u'(u'' - \frac{2f'(u')^2}{2}) \]

the first geodesic equation!
Thus unless \( u'(t) = 0 \), our curve satisfies both geodesic equations. ∎

Corollary. A geodesic on a surface of revolution asymptotic to the axis reaches \( z = -\infty \) \( \iff \) it is a meridian.

Proof. \( r \cos \phi < r \), so if \( r \to 0 \) as \( z \to -\infty \), we eventually reach some \( \Gamma \) = the constant for the geodesic. At this point \( \cos \phi = 1 \), so \( \alpha \) is (briefly) tangent to the parallel.
We now show that a geodesic has (locally) shortest length:

As before, we take a parametrized curve \( x(t) = (u(t), v(t)) \) and consider the variational vector fields

\[
\frac{d}{dx} \alpha_x(t) \bigg|_{x=0} = W(t), \quad W(0) = W(1) = \delta.
\]

But now the functional is

\[
\text{Length} = \int_0^1 \sqrt{E(u(t), v(t)) u'(t)^2 + 2Fu'v' + G(v')^2} \, dt
\]

Taking the \( x \) derivative and assuming \( x(t) \) was parametrized so this \( = 1 \),
we have to be careful about the chain rule

\[
\frac{d}{dx} \left( E(u_x(t),v_x(t)) \left( \frac{\partial}{\partial t} u_x(t) \right)^2 + 2 F(u_x(t),v_x(t)) \frac{\partial}{\partial t} u_x(t) \frac{\partial}{\partial x} v_x(t) + G(u_x(t),v_x(t)) \left( \frac{\partial}{\partial x} v_x(t) \right)^2 \right)^{1/2} =
\]

\[
\frac{1}{\alpha} \left( \text{the whole thing} \right)^{-1/2} \left( \langle [E_u, [E_v, \frac{\partial u}{\partial x}]] \rangle \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \right)
\]

\[
+ E \cdot 2 \left( \frac{\partial}{\partial x} u \right) \frac{\partial^2}{\partial x^2} u + 2 \left( \left[ \frac{\partial u}{\partial x} \right] \frac{\partial^2}{\partial x^2} u \right) \frac{\partial u}{\partial x} \frac{\partial u}{\partial x}
\]

\[
+ 2 \left( \frac{\partial^2}{\partial x^2} u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2}{\partial x^2} v \right) + \left( [G_u, [G_v, \frac{\partial v}{\partial x}]] \right) \frac{\partial v}{\partial x}
\]

\[
+ \alpha \left( \text{the whole thing} \right) = 1. \quad \text{Further,}
\]

\[
\left( \frac{\partial}{\partial x} u, \frac{\partial}{\partial x} v \right) = \hat{W} = (W_u, W_v)
\]
so we get

\[ \frac{1}{2} \left< u'(t)^2 \left[ \frac{E_u}{E_v} \right], \hat{W} \right> \]

\[ + E u'(t) W_u^1 \right\uparrow + \left< u'v' \left[ \frac{F_u}{F_v} \right], \hat{W} \right> \]

\[ + F(v' W_u + u' W_v) \]