Classification of points, Meusnier's formula

Last time, we revisited the classification of quadratics into ellipses, parabola, and hyperbola.

Definitions. If

- $K = \det S_p > 0$, $p$ is an **elliptic point**
- $K = \det S_p = 0$, $p$ is a **parabolic point**
- $K = \det S_p < 0$, $p$ is a **hyperbolic point**

It's also useful to name some other cases.

- $K_1 = K_2$, $p$ is an **umbilic point**
- $K_1 = K_2 = 0$, $p$ is a **planar point**
Claim. On a torus of revolution, the top and bottom planar circles are composed of parabolic points.

On a circle point, the normal does not change along the circle, so if \( V \) is tangent to the circle,

\[
S_p(V) = \sum_D V_n z = 0.
\]

On the other hand, all other slice curves bend in the same direction,
so all slice curvatures are \( \geq 0 \).
Since 0 is an extreme value for slice curvature, it is a principal curvature.

We now revisit slice curvatures.

![Diagram showing slice curve and normal vectors]

Proposition. If \( \alpha \) is any curve in \( M \) with unit tangent vector \( V \) at \( p \), then

\[
\Pi_p (v, v) = \langle xN, \hat{n} \rangle = k \cos \varphi
\]

where \( \varphi \) is the angle between \( \hat{n} \) and \( N \). We call this the normal curvature of \( \alpha \) at \( p \), denoted \( k_n \).
Proof. We've actually already done this computation, but to refresh.

\[ II_p(\vec{v}, \vec{v}) = \langle -D_v \hat{n}, \vec{v} \rangle \]

\[ = \langle -\hat{n}(\alpha(s)), T(s) \rangle \]

\[ = \langle \hat{n}(\alpha(s)), T'(s) \rangle \]

\[ = \langle \hat{n}, \kappa N \rangle. \quad \square. \]

**Examples.**

If \( \alpha \) is an asymptotic curve, \( X_n = 0 \) along \( \alpha \), so \( \langle N, \hat{n} \rangle = 0 \). This is true for all plane curves.

If two curves \( \alpha, \beta \) in \( M \) share the same tangent vector at \( p \), they have the same \( X_n \), even though their curvatures may be very different.
We now consider the general case of a surface of revolution.

\[ X(u,v) = (f(u) \cos v, f(u) \sin v, g(u)) \]

We now run through the computations.

\[ X_u = (f'(u) \cos v, f''(u) \sin v, g'(u)) \]
\[ X_v = (-f(u) \sin v, f(u) \cos v, 0) \]
\[ n = (-g'(u)f(u) \cos v, -g'(u)f(u) \sin v, f'(u)f(u)) \]
\[ \sqrt{f'(u)^2 g'(u)^2 + f''(u)^2 f(u)^2} \]
\[ (-g'(u) \cos v, -g'(u) \sin v, f'(u)) \]

since \( f'(u)^2 + g'(u)^2 = 1 \) (as arc length parametrized)

To continue,

\[
\begin{align*}
X_{uu} &= (f''(u) \cos v, f''(u) \sin v, g''(u)) \\
X_{uv} &= (-f'(u) \sin v, f'(u) \cos v, 0) \\
X_{vv} &= (-f(u) \cos v, -f(u) \sin v, 0)
\end{align*}
\]

So we have

\[
E = \langle x_u, x_u \rangle = f'(u)^2 \cos^2 v + f'(u)^2 \sin^2 v + g'(u)^2 \\
= f'(u)^2 + g'(u)^2 = 1.
\]

\[
F = \langle x_u, x_v \rangle = -f'(u)f(u) \cos v \sin v + f'(u)f(u) \cos v \sin v \\
= 0.
\]

\[
G = \langle x_v, x_v \rangle = f(u)^2 \sin^2 v + f(u)^2 \cos^2 v = f(u)^2.
\]

and

\[
E = \langle n, \dot{x}_{uu} \rangle = -g'(u)f''(u) \cos^2 v + g'(u)f''(u) \sin^2 v \\
+ f'(u)g''(u) \\
= f'(u)g''(u) - g'(u)f''(u)
\]
\[ m = \left< \vec{n}, \dot{X}_w \right> = f'(u)g'(u) \cos v \sin v - g'(u)f'(u) \sin v \cos v + O = 0. \]

\[ n = \left< \vec{n}, X_{ww} \right> = g'(u)f(u) \cos^2 v + g'(u)f(u) \sin^2 v + O = f(u)g'(u). \]

Here we can write out the shape operator

\[ S_p = \left[ \begin{array}{cc} E & F \\ F & G \end{array} \right]^{-1} \left[ \begin{array}{cc} \ell & m \\ m & n \end{array} \right] = \left[ \begin{array}{cc} G-F & \ell \\ -F & E \end{array} \right] \left[ \begin{array}{cc} \ell & m \\ m & n \end{array} \right] \cdot \frac{1}{EG-F^2} \]

\[ = \frac{1}{f^*(u)^2} \left[ \begin{array}{cc} f(u)^2 & 0 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} f'(u)g''(u) - g'(u)f''(u) & 0 \\ 0 & f^*(u)g'(u) \end{array} \right] \]

\[ = \left[ \begin{array}{cc} f'(u)g''(u) - g'(u)f''(u) & 0 \\ 0 & \frac{g'(u)}{f(u)} \end{array} \right] \]

This is already diagonalized, so we see

\[ K_1 = f'(u)g''(u) - g'(u)f''(u) \quad K_2 = \frac{g'(u)}{f(u)} \]

The principal directions are just \( X_w, X_v \).
To simplify, we first observe
\[ f'(u)^2 + g'(u)^2 = 1, \quad \text{so} \]
\[ 2f'(u)f''(u) + 2g'(u)g''(u) = 0. \]

So
\[
K = K_1 K_2 = \frac{1}{f(u)} \left[ f'(u) g'(u) g''(u) - g'(u)^2 f''(u) \right]
\]
\[
= -\frac{1}{f(u)} \left[ f'(u)^2 f''(u) + g'(u)^2 f''(u) \right]
\]
\[
= -\frac{f''(u)}{f(u)}.
\]

Example. The sphere of radius \( a \) is a surface of revolution with
\[
\alpha(u) = \left( a \cos \left( \frac{u}{a} \right), a \sin \left( \frac{u}{a} \right) \right).
\]

so \( f(u) = a \cos \left( \frac{u}{a} \right) \), \( f'(u) = -\frac{\sin \left( \frac{u}{a} \right)}{a} \),
and \( f''(u) = -\frac{1}{a} \cos \left( \frac{u}{a} \right) \). Thus
\[
K = -\frac{f''(u)}{f(u)} = -\frac{\cos \left( \frac{u}{a} \right)}{a \cos \left( \frac{u}{a} \right)} = \frac{1}{a}.
\]
Example. The surface of revolution of the tractrix is called the pseudosphere.

\[ \alpha(t) = (\tanh^2 t, 0, \text{sech} t). \]

This is not an arclength parametrization, so we compute

\[ \alpha'(t) = (1 - \text{sech}^2 t, 0, \text{sech}^2 t \tanh t) \]

\[ |\alpha'(t)|^2 = 1 - 2\text{sech}^2 t + \text{sech}^4 t + \text{sech}^2 t \tanh^2 t. \]

Now

\[ \cosh^2 t - \sinh^2 t = 1, \]

so

\[ 1 - \tanh^2 t = \text{sech} t \quad \text{or} \quad 1 - \text{sech}^2 t = \tanh^2 t. \]

Writing everything in \( \text{sech} t \),

\[ |\alpha'(t)|^2 = 1 - 2\text{sech}^2 t + \text{sech}^4 t + \text{sech}^2 t \tanh^2 t \]

\[ = 1 - \text{sech}^2 t = \tanh^2 t. \]
We now compute $K = -\frac{f''(s)}{f(s)}$

$f(t) = \text{sech} \ t$

so

$f'(s) = f'(t) \cdot \frac{dt}{ds} = f'(t) / \frac{ds}{dt}$

$= f'(t) = \frac{\text{sech} \ t \ \tanh t}{\tanh t} \frac{\tanh t}{\tanh t}$

$= \text{sech} \ t$.

Of course, this means that $f''(s) = \text{sech} \ t$

by the same argument. Thus

$K = -\frac{f''(s)}{f(s)} = -\frac{\text{sech} \ t}{\text{sech} \ t} = 1$.

Example. The plane has constant normal vector so $K = K_1 K_2 = 0$. 
We have now introduced 3 fundamental examples (we call these the space forms)

1) the sphere, $K = 1$
2) the plane, $K = 0$
3) the pseudosphere, $K = -1$.

We will return to these later on!