Codazzi and Gauss Equations II

Things now start to get heavy:

\[ S_p = \left[ \begin{array}{c} E & F \\ G & \end{array} \right]^{-1} \left[ \begin{array}{c} l \\ m \\ n \end{array} \right] = \frac{1}{EG-F^2} \left[ \begin{array}{c} G & F \\ -F & E \end{array} \right] \left[ \begin{array}{c} l \\ m \\ n \end{array} \right] \]

\[ = \frac{1}{EG-F^2} \left[ \begin{array}{cc} EG-mF & mG-nF \\ mE-lF & nE-mF \end{array} \right] = \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] \]

we can use this to compute

\[ \Pi_u = D_{x_u} n = -S_p (\hat{x}_u) = \frac{1}{EG-F^2} \left( (l_2 G-mF) x_u + (mE-lF) x_v \right) \]

\[ \Pi_v = D_{x_v} n = -S_p (\hat{x}_v) = \frac{1}{EG-F^2} \left( (mG-nF) x_u + (nE-mF) x_v \right) \]

We're in it to win it, so we return to this defining equations for the Christoffel symbols

\[ X_{uu} = \Gamma_{uu}^u x_u + \Gamma_{uu}^v x_v + \ell \hat{n} \]

\[ X_{uv} = \Gamma_{uv}^u x_u + \Gamma_{uv}^v x_v + m \hat{n} \]
and observe that

\[
X_{uvw} = (\Gamma^u_{uv})_v X_u + \Gamma^u_{uu} (\Gamma^v_{uv} X_u + \Gamma^v_{uu} X_v + m \vec{n}) \\
+ (\Gamma^v_{uw})_v X_v + \Gamma^v_{uv} (\Gamma^u_{uw} X_u + \Gamma^u_{uv} X_v + n \vec{n}) \\
+ l_v \vec{n} + l \vec{n} (-c X_u - d X_v) \\
\]

Collecting this in \( X_u, X_v, \) and \( n, \) we have

\[
= ((\Gamma^u_{uv})_v + \Gamma^u_{uu} \Gamma^u_{uu} + \Gamma^v_{uv} \Gamma^u_{uu} - c l) X_u \\
+ ((\Gamma^v_{uw})_v + \Gamma^v_{uu} \Gamma^v_{uu} + \Gamma^v_{uv} \Gamma^v_{uu} - d l) X_v \\
+ (\Gamma^u_{uu} m + \Gamma^v_{uv} n + l_v) \vec{n} \\
\]

Now this is equal to \( X_{uvu}, \) which is

\[
X_{uvu} = (\Gamma^u_{uv})_u X_u + \Gamma^u_{uv} (\Gamma^v_{uv} X_u + \Gamma^v_{uu} X_v + l \vec{n}) \\
+ (\Gamma^v_{uw})_u X_v + \Gamma^v_{uv} (\Gamma^u_{uw} X_u + \Gamma^u_{uv} X_v + m \vec{n}) \\
+ m_u \vec{n} + m (-a X_u - b X_v) \\
\]
As you'd expect, we collect these in $x_\mu, x_\nu, \hat{n}$:

$$X_{\mu} = \left( (\Gamma_\mu^\mu)_{\mu} + \Gamma_\mu^u \Gamma_\mu^u + \Gamma_\mu^v \Gamma_\mu^u - a m \right) X_\mu$$

$$+ \left( (\Gamma_\mu^v)_{\mu} + \Gamma_\mu^u \Gamma_\mu^v + \Gamma_\mu^v \Gamma_\mu^v - b m \right) X_\nu$$

$$+ \left( \Gamma_\mu^u l + \Gamma_\mu^v m + m_\mu \right) \hat{n}$$

Equating the coefficients, we have three equations. We can solve these to isolate the $l, m, n$ and $a, b, c, d$ variables:

$$l c - m a = (\Gamma_\mu^u)_{\nu} + \overbrace{\Gamma_\mu^u \Gamma_\mu^u}^{\text{understood}} + \Gamma_\mu^v \Gamma_\mu^u$$

$$- (\Gamma_\mu^u)_{\mu} - \overbrace{\Gamma_\mu^u \Gamma_\mu^u}^{\text{understood}} - \Gamma_\mu^v \Gamma_\mu^u$$

$$l d - m b = (\Gamma_\mu^v)_{\mu} + \Gamma_\mu^u \Gamma_\mu^v + \Gamma_\mu^v \Gamma_\mu^v$$

$$- (\Gamma_\mu^v)_{\nu} - \overbrace{\Gamma_\mu^v \Gamma_\mu^v}^{\text{understood}} - \Gamma_\mu^v \Gamma_\mu^v$$

$$l v - m u = \Gamma_\mu^u l + (\Gamma_\mu^v - \Gamma_\mu^u) m + \Gamma_\mu^v n$$
Returning to our first computations,

\[ \ell d - mb = \left( \ell (nE - mF) - m(mE - \ell F) \right) \frac{1}{EG-F^2} \]

\[ = \left( \ln E - \frac{m\ell F - m^2E + m\ell F}{EG-F^2} \right) \frac{1}{EG-F^2} \]

\[ = E \left( \frac{\ln - m^2}{EG-F^2} \right). \]

But we know

\[ K = \det S_p = \det \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} \]

\[ = \frac{\det \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}}{\det \begin{bmatrix} E & F \\ F & G \end{bmatrix}} = \frac{\ln - m^2}{EG-F^2}. \]

Thus we have shown

\[ EK = \langle \text{some mess of Christoffel symbols and their derivatives!} \rangle \]

This is the first Gauss equation.
Theorem. (Gauss's Theorema Egregium)
$I_p$ determines Gauss curvature.

Proof. Already done - the Christoffel symbols determine $K$ and $I_p$ determines the Christoffel symbols. □

Now you'll note that we have three more equations with $\ell, m, n$ and $a, b, c, d$ and two equations with partials of $\ell, m, n$ (from matching $n$ components) if we also use $X_{\mu\nu\rho} = X_{\nu\mu\rho}$.

It turns out that the three give us equations for $FK$ and $GK$:

$$FK = (\Gamma_{\mu\nu}^u)_u - (\Gamma_{\mu u}^u)_v + \Gamma_{\mu v}^u \Gamma_{u\nu}^u - \Gamma_{\mu u}^v \Gamma_{u\nu}^u$$

$$FK = (\Gamma_{\mu\nu}^v)_v - (\Gamma_{\nu \mu}^v)_u + \Gamma_{\nu \mu}^v \Gamma_{\mu\nu}^v - \Gamma_{\mu \nu}^u \Gamma_{\mu\nu}^u$$

$$GK = (\Gamma_{\nu\mu}^u)_u - (\Gamma_{\mu\nu}^u)_v + \Gamma_{\mu\nu}^u \Gamma_{\nu\mu}^u + \Gamma_{\nu\mu}^v \Gamma_{\mu\nu}^u - \Gamma_{\mu\nu}^u \Gamma_{\nu\mu}^u \Gamma_{\nu\mu}^u$$

These are also called Gauss equations.
The remaining two are called Codazzi equations:

\[\ell_v - m_u = \ell \Gamma^u_{wv} + m(\Gamma^v_{uw} - \Gamma^w_{uv}) + n \Gamma^v_{uw}\]

\[m_v - n_u = \ell \Gamma^u_{wv} + m(\Gamma^v_{uw} - \Gamma^w_{uv}) + n \Gamma^v_{uw}\]

If we're really pressed, we can use the Gauss equations to compute Gauss curvature. But the main point was to show the Theorema Egregium.

Corollary. If \( M \) and \( M^* \) are locally isometric, their Gauss curvatures \( K \) and \( K^* \) are equal.

We say this means

"Gauss curvature is an isometry invariant."

Since this may be the first time
you've seen an invariant, let's unpack what this means.

\[ M \text{ isometric to } M^* \Rightarrow K = K^* \]
is the same as

\[ K \neq K^* \Rightarrow M \text{ not isometric to } M^* \]

This can be quite powerful.

Example. No map of the Earth can have correct angles, lengths and areas.

Proof. The map is on a planar sheet of paper with \( K = 0 \) and the curvature \( K^* \) of the sphere is 1. Hence \( I_p \) (map) is different from \( I_p^* \) (earth). But that means lengths and angles are different on map and Earth (if not, we could use

\[ I_p(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = |\tilde{\mathbf{u}}||\tilde{\mathbf{v}}| \cos \Theta = |\tilde{\mathbf{u}}_x||\tilde{\mathbf{v}}_x| \cos \Theta = I_{p^*}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \]

to show \( I_p \) (map) = \( I_p^* \) (earth). )
It is natural to ask whether
\[ K = K^* \implies M \text{ isometric to } M^* \]
even though this more powerful statement does not follow from
\[ M \text{ isometric to } M^* \implies K = K^*. \]

**Example.**

Answer: No. You’ll do an example in homework, but the idea is that ellipsoids
are really not isometric.

\[ k_2 = \frac{1}{2} \quad \text{and} \quad k_3 = 1 \]

\[ k_2 = 2 \]

Slightly bogus.