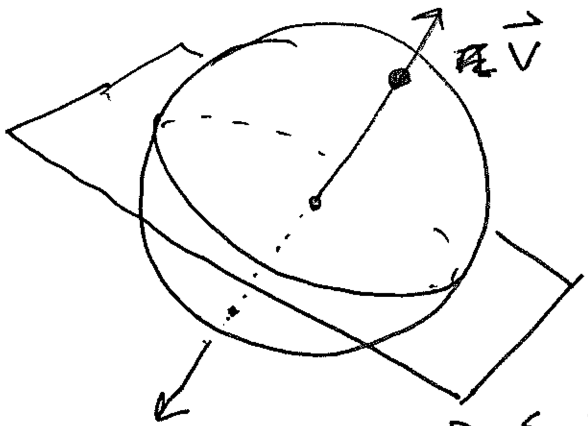
Crofton's Formula and Buffon's Needle.

We now want to move from projections to intersections. Oddly, it's easier to start on the sphere.

Definition. The ~~space~~ intersection of a plane through the origin with a sphere is called a great circle.

Definition. The space of planes through the origin in $\mathbb{R}^3$ is called the "Grasmann manifold" $G_2(\mathbb{R}^3)$.

We can parametrize $G_2(\mathbb{R}^3)$ by the sphere itself (but it's a 2-1 map).



$P = \{\text{plane with normal } \vec{v}\}$

Notice that $\vec{v}$ and $-\vec{v}$ encode the same plane, and that $G_2(\mathbb{R}^3)$ is a 2 dimensional space.

Definition. We measure area on $G_2(\mathbb{R}^3)$ by spherical area, and define the average value of a function $f(P)$ by

$$\mathrm{Average}_{P \in G_2(\mathbb{R}^3)} f(P) = \frac{1}{4\pi} \int_{\vec{n} \in S^2} f(P_{\vec{n}}) \, d\mathrm{Area}_{S^2}.$$

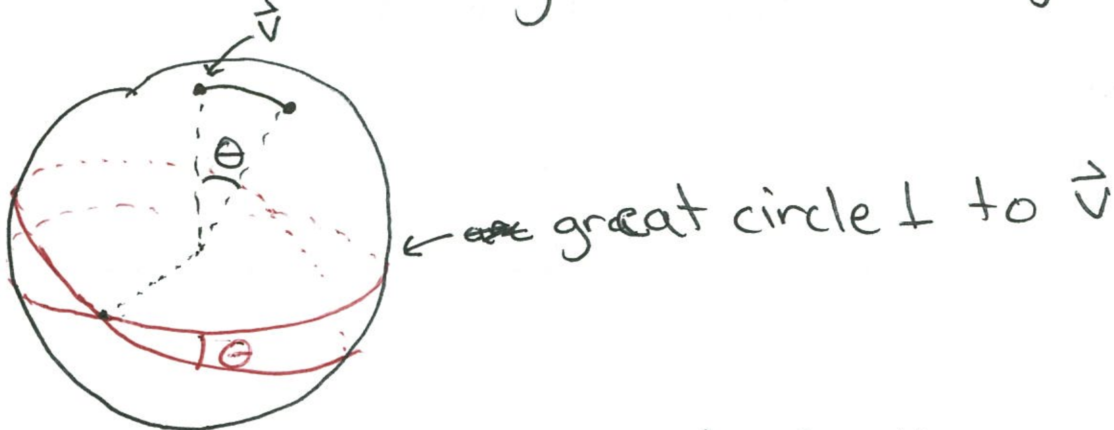
Important: Rigid rotations (matrices in $SO(3)$) transform planes and normals in the same way, and preserve spherical and Grassmannian area.

Now we can prove

Theorem (Crofton's Formula)
The length of a spherical curve γ is given by $\pi \, \text{Average}_{P \in G_2(\mathbb{R}^3)} \begin{pmatrix} \#\text{ of intersections} \\ \text{of } \gamma \text{ with } P \end{pmatrix}$

Proof. We are going to approximate γ by a spherical polygon made up of arcs of great circles.

Start with a single arc of length θ.



If we start with a point $\vec{v}$, the great circles through $\vec{v}$ have normals along the great circle $\perp$ to $\vec{v}$. Pushing $\vec{v}$ through an angle θ sweeps out a lune of angle θ.

Fun fact. The area of the lune is
$4\pi \cdot \frac{\theta}{\pi} = 4\theta$, as it covers $\frac{\theta}{\pi}$ of the entire sphere.

Note: We used rotational invariance to conclude that this computation is the same for every great circle segment.

So for this segment,

Length $\gamma = \theta$

$\text{Average}_{P \in G_2(\mathbb{R}^3)}$ # of intersections =

$$= \frac{1}{4\pi} \int_{\vec{n} \in S^2} \begin{cases} 1, & \text{if } \vec{n} \text{ is in the lune} \\ 0, & \text{if } \vec{n} \text{ is not in the lune} \end{cases} \, d\text{Area}$$

$$= \frac{1}{4\pi} \cdot \text{Area of lune} = \frac{1}{4\pi} \cdot 4\theta = \frac{\theta}{\pi}.$$

and

$$\text{Length } \gamma = \pi \cdot \text{Average}_{P \in G_2(\mathbb{R}^3)} \# \text{ of intersections}.$$

Now observe that if we add segments, both left and right ~~are~~ & functions add.

and finally conclude the theorem for any rectifiable spherical curve by approximation. □

Example. A great circle crosses every other great circle twice, so has length 2π.

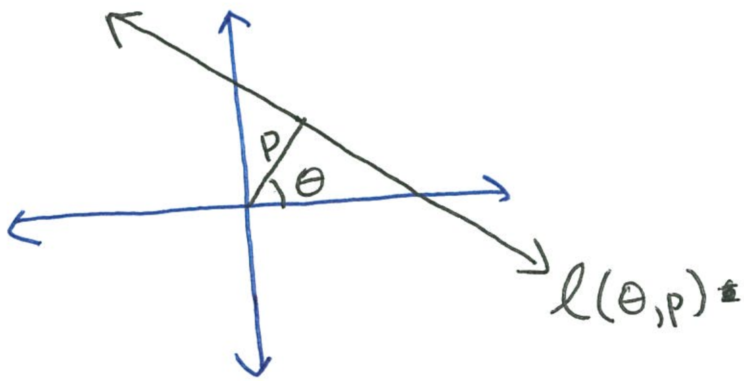
Fenchel's Theorem. The total curvature of a closed space curve is at least 2π.

Proof. We already showed T crosses every great circle (at least twice, since T is a closed curve).

Now let's try to do this theorem in the plane.

So we need to parametrize the space of lines in $\mathbb{R}^2$.

Once again, this is a two-dimensional space

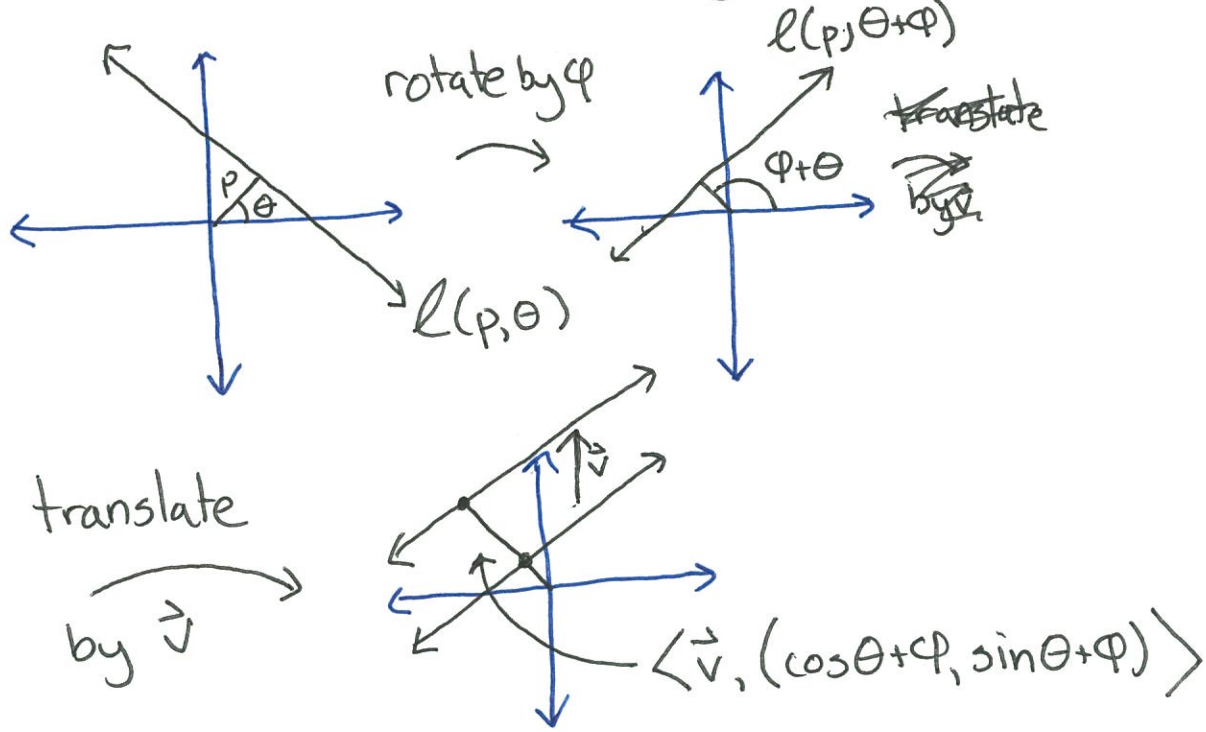


Definition. $\ell(p,\theta)$ is the line
$$(\cos\theta)x + (\sin\theta)y = p.$$

We will integrate over the space of lines by integrating $d\theta\, dp$.

Proposition. A rigid motion of $\mathbb{R}^2$ preserves the area $\int dp\, d\theta$ on lines.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a rigid motion consisting of rotation by φ and translation by $\vec{v}$. The corresponding map on lines



is given by

$$f(p,\theta) = \left(p + \langle \vec{v}, (\cos(\theta+\varphi), \sin(\theta+\varphi)) \rangle, \theta+\varphi\right)$$

By the change of variables formula

$$\int_{f(U)} dp\,d\theta = \int_U |\det Df|\, dp\,d\theta$$

for any ~~collec~~ set of lines $\mathcal{U}$. So we compute

$$\det Df = \begin{vmatrix} 1 & \langle \vec{v}, (\sin(\theta+\varphi), \cos(\theta+\varphi)) \rangle \\ 0 & 1 \end{vmatrix}$$

$$= 1,$$

which proves the theorem. □

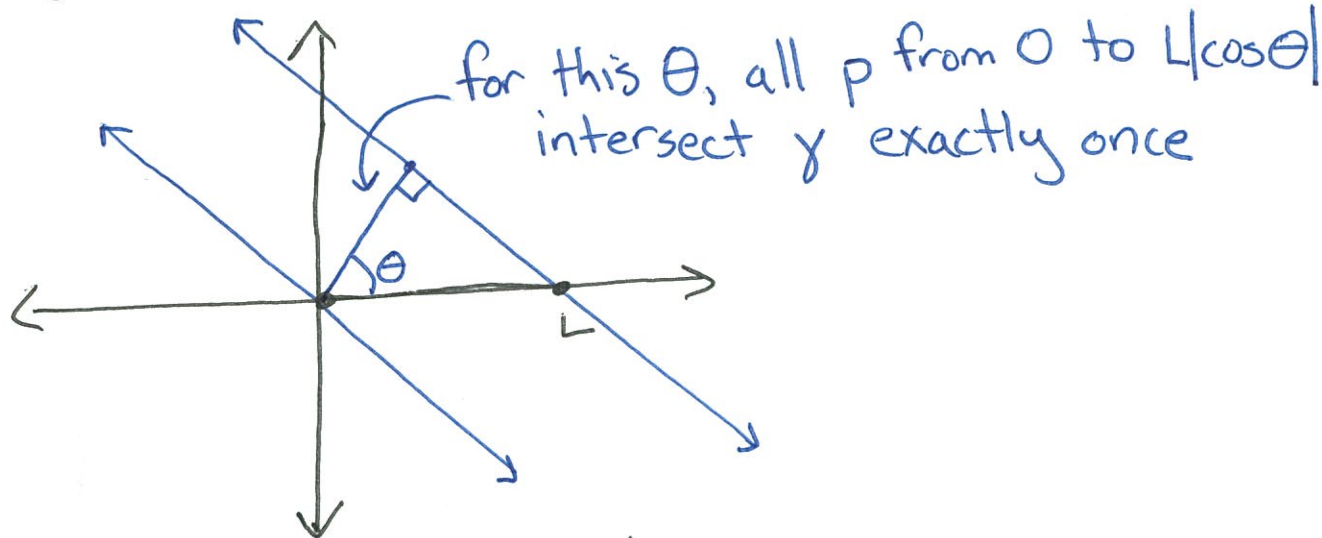
We can now show

Theorem. For any plane curve, γ,

$$\text{Length } \gamma = 4 \int \#\begin{smallmatrix}\text{intersections of} \\ \gamma \text{ and } \ell(p,\theta)\end{smallmatrix} \, dp\, d\theta.$$

Proof. Because both sides add if we combine curves, ~~solvin~~ proving the formula for a line segment proves it for all polygons, and (by approximation) for all curves.

Since $dp\,d\theta$ is invariant under rigid motions, wlog the segment might as well be $(0,L)$ along x-axis.

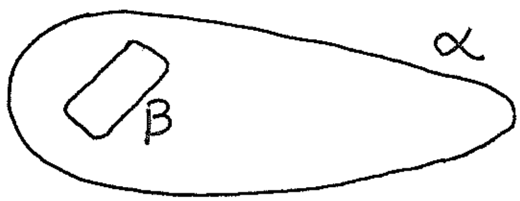


So we are integrating

$$\int_0^{2\pi} L|\cos\theta|\,d\theta = 4L. \qquad \square$$

We can use this to prove a cute corollary.

Proposition. Let α and β be convex plane curves with $\beta \subset \alpha$. The probability that a random line intersecting α also intersects β is $\text{Length}(\beta)/\text{Length}(\alpha)$.



Proof. We know that every line intersects a convex curve 0 or 2 times, so

$$\frac{\text{Volume (lines intersecting } \beta)}{\text{Volume (lines intersecting } \alpha)} = \frac{\frac{1}{2}\int \#\, \ell(p,\theta)\cap\beta \; dp\, d\theta}{\frac{1}{2}\int \#\, \ell(p,\theta)\cap\alpha \; dp\, d\theta}$$

$$= \frac{\cancel{\tfrac{1}{8}}\ \text{Length}(\beta)}{\cancel{\tfrac{1}{8}}\ \text{Length}(\alpha)} =$$

Now <u>every</u> line through β intersects α, so this quotient is the probability above. $\square$