Crofton's Formula and the indicatrices.

We now know that for a curve on S^2,

$$\text{Length} = \pi \times \text{Average}_{P \in G_2(\mathbb{R}^3)} \#\,\text{intersections with } P$$

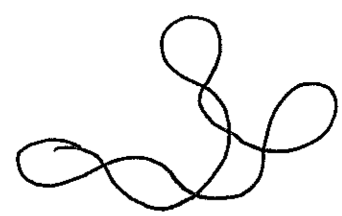
we are going to take this out for a spin!

Definition*: A curve γ in $\mathbb{R}^3$ is a nontrivial knot if $\nexists$ a smooth map $f: D^2 \to \mathbb{R}^3$ so that f is 1-1 and $f|_{\partial D^2}: S^1 \to \mathbb{R} = \gamma$.

Examples.

Knot

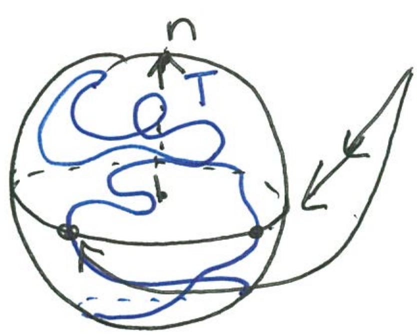
not a Knot



not a Knot either!

Theorem [Fáry-Milnor]
If γ is knotted, it has total curvature $\geq 4\pi$.

Proof. We argue by proving the contrapositive. Suppose the total curvature of $\gamma < 4\pi$. Then the tangent indicatrix has length $< 4\pi$, and by Crofton's formula must cross some plane only twice.



If $\vec{n}$ is the normal to the plane, $T \cdot \vec{n} = 0$ only at these points.

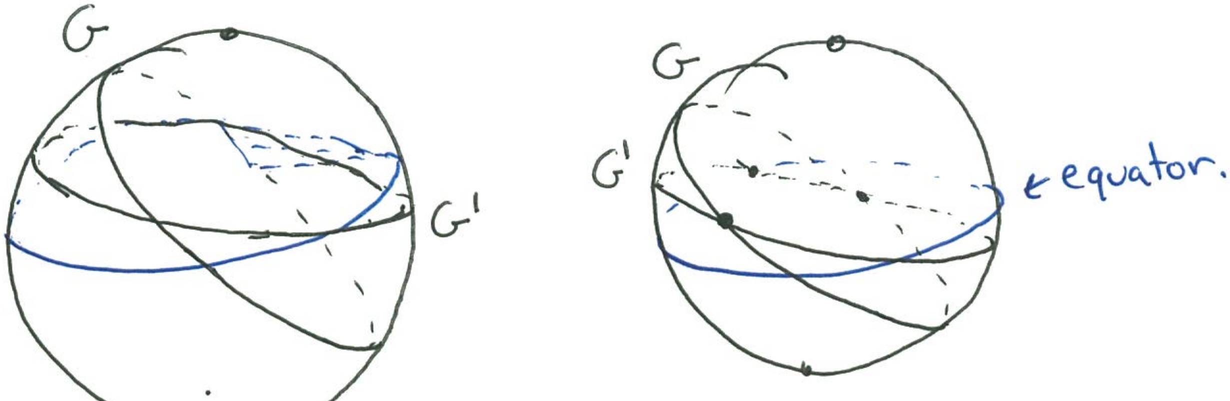
Consider the function $\langle \vec{\gamma}(s), \vec{v} \rangle = f(s)$. We see $f'(s) = \langle T(s), \vec{v} \rangle$ so f has only two critical points. This means that f is increasing on one arc of S^1 and decreasing on the other.

γ thus intersects any plane with normal $\vec{v}$ between $\min\langle \gamma, \vec{v} \rangle$ and $\max\langle \gamma, \vec{v} \rangle$ exactly twice. Joining these points with a line creates the disk. $\square$

Now we'll try something harder.

Theorem [Milnor-Totaro].
If γ is closed and $\tau(s) \geq 0$, while $\tau(s)$ is not always zero, then
$$\int \sqrt{\kappa^2 + \tau^2}\, ds = \text{Length } N(s) \geq 4\pi.$$

To prove it, we'll need a lemma



~~L~~Lemma. Suppose we have two great circles G and G', neither including the north or south poles.

~~An~~ Half of G' is south of G, and half north. Call the southern arc G'_S. If the tangent vector along G' pointing from G'_N to G'_S ~~points~~ lies in the northern hemisphere, then G' makes a smaller angle with the equator than G.

Proof. This tangent vector points along ~~the~~ G' in the direction from the southern-most point to the northernmost (since it is pointing northward). Hence the southernmost point of G' is north of G.

Next, recall that we have shown that if we think of $T(s)$ as $\tilde{\gamma}(\tilde{s})$,

then $\tilde{\gamma}(\tilde{s})$ is a spherical curve with

$$\tilde{T}(\tilde{s}) = N(s)$$

$$\begin{aligned}\tilde{T}'(\tilde{s}) &= -\kappa(s)T(s)\cdot\frac{ds}{d\tilde{s}} + \tau(s)B(s)\frac{ds}{d\tilde{s}}\\ &= -T(s) + \frac{\tau(s)}{\kappa(s)}B(s).\end{aligned}$$

In particular, if $\tau(s) > 0$, then the curve $\tilde{\gamma}$ is always heading in the $N(s)$ direction and turning towards $B(s)$.

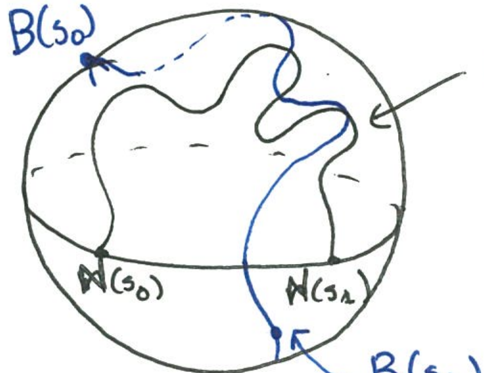
Proof (of Theorem).

~~We argue by contradiction.~~

We claim $N(s)$ intersects every great circle at least 4 times.

Suppose not. Wlog, we may take the great circle crossed twice to be the equator, and suppose the crossings to be at s_0, s_1.

Let $\varphi(\vec{v})$ denote angle above the equator (latitude) with $-\pi/2 \leq \varphi \leq \pi/2$.



We choose $s_0 < s < s_1$ so that $\varphi(N) > 0$ on this range, <u>and</u> so

$$|\varphi(B(s_0))| \geq |\varphi(B(s_1))|.$$

(We might have to ~~change of the~~ swap north and south or reverse the parametriz. of N, but that's ok.)

There is a great circle tangent to T at $T(s_0)$, call it G.

Claim. T touches, but does not cross G.

(Note that if we prove this, we have found a contradiction!)

We restrict ourselves to $s_0 \leq s \leq s_1$.

Now consider $\varphi(B(s))$. Since $B'(s) = -\tau N(s)$, $B'(s)$ is pointing southwards (since N is in ~~pointing~~ northern hemisphere by assumption that $\varphi(N) > 0$ on this range).

So $\varphi(B)$ is decreasing on (s_0, s_1) and strictly decreasing when $\tau > 0$.

If $\varphi(B(s_0)) = \varphi(B(s_1))$, then $\tau = 0$ on the whole interval (and, flipping parametrization, on the rest of γ), which is ruled out by hypothesis.

So $\varphi(B(s_0)) > \varphi(B(s_1))$. Since ~~$|\varphi(B(s_0))| \geq$~~ also $|\varphi(B(s_0))| \geq |\varphi(B(s_1))|$, this means $\varphi(B(s_0)) > 0$ (it's ~~grea~~ more northerly and further away from equator)

Now suppose T crosses G at some s' with $s_0 < s' < s_1$. Let $T(s')$ be the first such crossing, and let G' be the great circle tangent to T at s'.

Claim. The lemma applies to G, G', E.

1) The normal to G is $B(s_0)$ which has $\varphi(B(s_0)) > 0$, so G does not pass through the poles.

2) Since T curves towards B, and at s_0, B is in the northern hemisphere, T is north of G until it crosses again at s', where it is crossing to the south.

But at this point, the tangent to T is G', which must also be crossing G from north to south. The tangent to T is N, ~~and~~ and $\varphi(N(s')) > 0$ by our setup above. Thus the tangent to G' points north, and G' makes a smaller angle with the equator than G.

But this means $\varphi(B(s')) > \varphi(B(s_0))$, contradicting our earlier proof that $\varphi(B)$ was ~~mono~~ decreasing! $\times\!\!\times$. $\square$

We don't have time to prove it, but Milnor gave another pretty theorem along these lines.

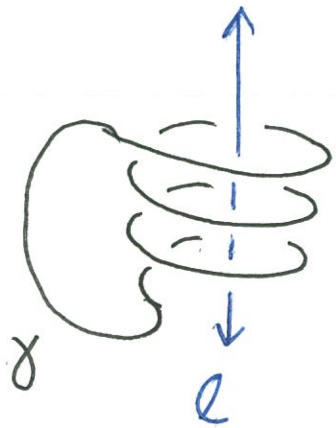
Definition. The linking number of a closed space curve γ with a line ℓ is the ~~oriented~~ integral

$$\frac{1}{2\pi}\int_S \Theta(\gamma(s))\,ds$$

where Θ is the angle in cylindrical coordinates with axis ℓ.

Theorem. If γ has linking number n with a line,

$$\int \kappa(s)\,ds + \int |\tau|\,ds \geq 2\pi n.$$



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