The Gauss Map and the Second Fundamental Form: measuring the curvature of a surface.

We are interested now in describing the geometry of a surface in a more compact way. Here’s the idea:

The surface bends the same way along each coordinate line. We can track that normal vector to the surface. We want to distinguish this from where the surface bends in opposite directions along each coordinate line.
Definition. A surface $S$ is orientable if there is a differential map $N:S \to S^2$ so that

$$N(p) \perp T_p S$$

at every point on $S$.

The map $N(p)$ is called the Gauss map.

We can give the Gauss map in local coordinates $\tilde{X}:U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Since $T_p S$ has the basis $\tilde{X}_u, \tilde{X}_v$ we know that

$$N(p) = \pm \frac{\tilde{X}_u \times \tilde{X}_v}{|\tilde{X}_u \times \tilde{X}_v|}$$
At any given point, there are two choices for the normal vector, so

\[ \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|} \]

agrees or disagrees with \( N \).

If this is \(-N\), we can "fix things" by changing the names of \( u \) and \( v \).

**Beware.** Not every regular surface is an orientable surface. We might not be able to patch all the local normal fields into a global one. A good example is the Möbius strip at left.
Since we have $N : S \to S^2$, 

we have 

$$dN_p : T_p S \to T_{N(p)} S^2$$

But $T_{N(p)} S^2$ is normal to $N(p)$, since it's a tangent plane to the sphere, and $T_p S$ is normal to $N(p)$, by definition. So there is a natural map 

$$T_{N(p)} S^2 \xrightarrow{\text{Id}} T_p S$$

and we can think of $dN_p$ as a map 

$$dN_p : T_p S \to T_p S.$$
Examples:

Computing $dN_p$ in local coordinates, and in terms of a consistent system of coordinates on $\mathbb{R}^3$.

Trick: Use $N_{ox}$ to define local coordinates on $S^2$. Then the map $N$ is just

$$N(u,v) = (u,v).$$

So $dN_p = I$. But on $T_{N(p)}S^2$, we are in the basis $N_u, N_v$ and on $T_pS$ we are in the basis $X_u, X_v$. 
So in $\mathbb{R}^3$ on $T_pS$, we have
\[ dN_p : T_pS \to T_pS \text{ given by } x_v \mapsto N_v. \]

Examples.

1. The $x$-$y$ plane.

\[ X(u,v) = (u,v,0), \quad N(u,v) = (0,0,1). \]
\[ X_u = (1,0,0), \quad N_u = (0,0,0) \]
\[ X_v = (0,1,0), \quad N_v = (0,0,0) \]

so
\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]
everywhere.

2. The unit sphere.

\[ X(u,v) = (u,v,\sqrt{1-u^2-v^2}), \quad \tilde{N} = \left( \frac{2u}{\sqrt{1-u^2-v^2}}, \frac{2v}{\sqrt{1-u^2-v^2}}, 1 \right) \]
\[ X_u = (1,0,\frac{-2u}{\sqrt{1-u^2-v^2}}), \quad N_u = \frac{\sqrt{1-u^2-v^2}}{1-u^2-v^2} \left( \frac{2u}{\sqrt{1-u^2-v^2}}, \frac{2v}{\sqrt{1-u^2-v^2}}, 1 \right), \]
\[ X_v = (0,1,\frac{-2v}{\sqrt{1-u^2-v^2}}) \]
We then have

\[ X_u \times X_v = \left( \frac{U}{\sqrt{1-u^2-v^2}}, \frac{V}{\sqrt{1-u^2-v^2}}, 1 \right). \]

\[ = \frac{1}{\sqrt{1-u^2-v^2}} \left( U, V, \sqrt{1-u^2-v^2} \right) \]

so

\[ N = (U, V, \sqrt{1-u^2-v^2}) = X \]

and

\[ N_u = \frac{\partial X}{\partial u} \]

\[ N_v = \frac{\partial X}{\partial v} \]

so we have

\[ dN_p : T_p S - T_p S \]

\[ = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \]
③ The Hyperbolic Paraboloid

\[ \vec{X}(u, v) = (u, v, v^2 - u^2) \]

We compute

\[ X_u = (1, 0, -2u) \]
\[ X_v = (0, 1, 2v) \]

\[ X_u \times X_v = (2u, -2v, 4) \]

\[ |X_u \times X_v| = \sqrt{1 + 4u^2 + 4v^2} = 2\sqrt{1 + u^2 + v^2} \]

so

\[ N = \begin{pmatrix} u \\ \frac{-v}{\sqrt{u^2 + v^2 + 1/4}} \\ \frac{1}{\sqrt{u^2 + v^2 + 1/4}} \end{pmatrix} \]

and we have

\[ N = \frac{1}{\sqrt{u^2 + v^2 + 1/4}} (u, -v, 1) \]

\[ N_u = \frac{1}{\sqrt{u^2 + v^2 + 1/4}} \left( u, v, 1 \right) + \frac{1}{\sqrt{u^2 + v^2 + 1/4}} (4, 0, 0) \]
\[ N_u = \frac{1}{\sqrt{v^2 + u^2 + \frac{1}{4}}} \left( 1 - \frac{u^2}{v^2 + u^2 + \frac{1}{4}}, \frac{uv}{v^2 + u^2 + \frac{1}{4}}, \frac{-2v}{2(v^2 + u^2 + \frac{1}{4})} \right) \]

\[ N_v = \frac{1}{\sqrt{v^2 + u^2 + \frac{1}{4}}} \left( \frac{-uv}{v^2 + u^2 + \frac{1}{4}}, -1 - \frac{v^2}{v^2 + u^2 + \frac{1}{4}}, \frac{-4v}{2(v^2 + u^2 + \frac{1}{4})} \right) \]

At \( u = v = 0 \), we have

\[ X_u = (1, 0, 0) \quad N_u = (1, 0, 0) = X_u \]

\[ X_v = (0, 1, 0) \quad N_v = (0, -1, 0) = -X_v \]

Therefore, we have

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

This one requires a little consideration. The columns of this matrix are \( N_u, N_v \), written in the \( X_u, X_v \) basis as

\[ N_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad N_v = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \]
We notice that in all 3 cases, $dN_p$ was a symmetric matrix. That's not a coincidence!

Proposition. For any vectors $\mathbf{v}, \mathbf{w} \in T_p S$, we have $\left< \mathbf{v}, \mathbf{dN}_p(\mathbf{w}) \right>_p = \left< \mathbf{dN}_p(\mathbf{v}), \mathbf{w} \right>_p$ (where the inner product is $I_p$: the inner product on $\mathbb{R}^3$).

Proof. Given a basis $X_u, X_v$ for $T_p S$, we only have to check

$$\left< \mathbf{X}_u, \mathbf{dN}_p(\mathbf{X}_v) \right> \overset{?}{=} \left< \mathbf{dN}_p(\mathbf{X}_u), \mathbf{X}_v \right>$$

as vectors in $\mathbb{R}^3$. Now this is really

$$\left< \mathbf{X}_u, \mathbf{N}_v \right> \overset{?}{=} \left< \mathbf{N}_u, \mathbf{X}_v \right>.$$

We know $\left< \mathbf{N}, \mathbf{X}_u \right> = 0$, so

$$\frac{\partial}{\partial u} \left< \mathbf{N}, \mathbf{X}_u \right> = \left< \mathbf{N}_u, \mathbf{X}_u \right> + \left< \mathbf{N}, \mathbf{X}_{uu} \right> = 0.$$

and similarly

$$\frac{\partial}{\partial v} \left< \mathbf{N}, \mathbf{X}_v \right> = \left< \mathbf{N}_u, \mathbf{X}_v \right> + \left< \mathbf{N}, \mathbf{X}_{uv} \right> = 0.$$
Thus we have

\[ \langle \dot{N}_v, \dot{X}_u \rangle = -\langle \dot{N}_u, \dot{X}_w \rangle = \langle \dot{N}_u, \dot{X}_v \rangle \]

as desired.

Comment. Does this mean that our previous matrix is symmetric? No!

Suppose

\[ N_0 = a X_u + c X_v \]
\[ N_v = b X_u + d X_v \]

so

\[ \frac{dN_p}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

If \( I_p \) is given by the matrix

\[ I_p = \begin{bmatrix} \langle X_u, X_u \rangle & \langle X_v, X_u \rangle \\ \langle X_u, X_v \rangle & \langle X_v, X_v \rangle \end{bmatrix} \]
then this means that the matrix

\[ I_p dN_p = \begin{bmatrix} b\langle X_u, X_0 \rangle + d\langle X_v, X_0 \rangle \\
0 & a\langle X_v, X_u \rangle + c\langle X_0, X_0 \rangle \end{bmatrix} \]

is symmetric. Or that

\[ a\langle X_u, X_v \rangle + c\langle X_v, X_u \rangle = \langle aX_u + cX_v, X_u \rangle = \langle N_u, X_v \rangle \]

and

\[ b\langle X_u, X_0 \rangle + d\langle X_v, X_0 \rangle = \langle bX_u + dX_v, X_0 \rangle = \langle N_v, X_0 \rangle, \]

which is exactly what we proved above.
This leads us to define:

Definition. The quadratic form

\[ \Pi_p (\tilde{\nu}, \tilde{\omega}) = - \langle dN_p (\tilde{\nu}), \tilde{\omega} \rangle \]

is called the second fundamental form of the surface \( S \) at \( p \).