The Gauss Theorem and Mainardi-Codazzi Equations

We know that $X_u, X_v, N$ form a basis for $\mathbb{R}^3$ at $p$. So we can write

\[
X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L_1 N
\]

\[
X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + L_2 N
\]

\[
X_{vu} = \Gamma^1_{21} X_u + \Gamma^2_{21} X_v + L_2 N
\]

\[
X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + L_3 N
\]

\[
N_u = a_{11} X_u + a_{21} X_v
\]

\[
N_v = a_{12} X_u + a_{22} X_v
\]

Now

\[
\langle X_{uu}, N \rangle = L_1 = e
\]

\[
\langle X_{uv}, N \rangle = L_2 = \langle X_{vu}, N \rangle = L_2 = f
\]

\[
\langle X_{vv}, N \rangle = L_3 = g
\]
What about the $\Gamma_{ij}^k$? These are called Christoffel symbols, and are defined by

$$\frac{\partial^2}{\partial u_i \partial u_j} x = \sum \Gamma_{ij}^k \frac{\partial}{\partial u_k} x + \text{stuff normal to } T_p S$$

In more advanced classes, you'll see this written

$$\frac{\partial^2}{\partial u_i \partial u_j} x = \sum_{ij} \Gamma_{ij}^k \frac{\partial x}{\partial u_k}$$

The notation that we infer a summation over repeated indices is called the Einstein convention, or Einstein summation convention.

(This may be the first time in your life you'll learn a piece of Einstein's mathematics—treasure it!)

Going back to work, we see
\[ \frac{1}{2} \frac{\partial}{\partial u} \langle x_u, x_u \rangle = \langle x_u, x_u \rangle \]

\[ = \Gamma^1_{n} \langle x_u, x_u \rangle + \Gamma^2_{n} \langle x_u, x_v \rangle + L_4 \langle x_u, N \rangle \]

\[ = \Gamma^1_{n} E + \Gamma^2_{n} F. \]

Or, we see that

\[ \Gamma^1_{n} E + \Gamma^2_{n} F = \frac{1}{2} E_u. \]

Now

\[ \langle x_u u, x_v \rangle = \Gamma^1_{n} F + \Gamma^2_{n} G \]

and

\[ \langle x_u u, x_v \rangle = \frac{\partial}{\partial u} \langle x_u, x_v \rangle - \langle x_u, x_u \rangle \]

\[ = F_u - \langle x_u, x_u \rangle \]

\[ = F_u - \frac{1}{2} E_v \]

So

\[ \Gamma^1_{n} F + \Gamma^2_{n} G = F_u - \frac{1}{2} E_v \]
Continuing,

\[ \langle X_{uv}, X_u \rangle = \Pi_{12}^4 E + \Pi_{12}^2 F = \frac{1}{2} E v \]

\[ \langle X_{uv}, X_v \rangle = \Pi_{12}^4 F + \Pi_{12}^2 G = \frac{1}{2} G_u \]

and

\[ \langle X_{uu}, X_u \rangle = \Pi_{22}^4 E + \Pi_{22}^2 F = F_v - \frac{1}{2} G_u \]

\[ \langle X_{uv}, X_v \rangle = \Pi_{22}^4 F + \Pi_{22}^2 G = \frac{1}{2} G_v \]

We could solve each of these three 2\times2 systems of equations for \( \Pi_{12}^4, \Pi_{12}^2, \Pi_{22}^4 \)

since each system is based on the matrix \[ \begin{bmatrix} E & F \\ F & G \end{bmatrix} \] which has determinant \( EG-F^2 \neq 0 \).

This wouldn't be hard. Observe

\[
\begin{bmatrix}
E & F \\
F & G
\end{bmatrix}\begin{bmatrix}
\Pi_{12}^4 \\
\Pi_{12}^2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} E u \\
F u - \frac{1}{2} E v
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\Pi_{12}^4 \\
\Pi_{12}^2
\end{bmatrix} = \frac{1}{EG-F^2} \begin{bmatrix}
G - F \\
-F & E
\end{bmatrix}\begin{bmatrix}
\frac{1}{2} E u \\
F u - \frac{1}{2} E v
\end{bmatrix}
\]
So, just to be specific in one case,

\[
\Gamma^1_{11} = \frac{1}{E - F^2} \left( \frac{1}{2} G E_u - F F_u + \frac{1}{2} F E_v \right),
\]

and the other \( \Gamma^k_{ij} \) are similar combinations of \( E, F, G \) and their partials.

Proposition. The Christoffel symbols are isometry invariants.

Proof. Isometries preserve the functions \( E, F, G \) (and hence their derivatives, etc.).

Example. Consider the surface of revolution

\[ X(u, v) = (\varphi(u) \cos u, \varphi(v) \sin u, \psi(v)) \]

We recall that

\[ E = \varphi'^2(u), \quad F = 0, \quad G = (\varphi'^2(u) + \psi'^2(v)) \]

so

\[ E_u = 0, \quad E_v = \varphi \varphi', \quad F_u = F_v = 0, \quad G_u = 0, \quad G_v = \frac{3}{2} \varphi' \varphi \]

(2)}
We then see that

\[ \Gamma^0_0 \varphi^2 + \Gamma^2_0 = \frac{1}{2} \cdot 0 \]

\[ \Gamma^0_0 + \Gamma^2_0 (\varphi^2 + \psi^2) = 0 - \frac{1}{2} \varphi \varphi' \]

So

\[ \Gamma^0_0 = 0, \quad \Gamma^2_0 = \frac{-\varphi \varphi'}{\delta(\varphi^2 + \psi^2)} \]

And

\[ \Gamma^1_0 \varphi^2 + \Gamma^2_0 = \frac{1}{2} \varphi \varphi' \]

\[ \Gamma^1_0 + \Gamma^2_0 (\varphi^2 + \psi^2) = \frac{1}{2} \cdot 0 \]

So

\[ \Gamma^1_0 = \frac{\varphi'}{2 \varphi}, \quad \Gamma^2_0 = 0 \]
And then we have

\[ \Gamma_{22}^1 \phi^2 + \Gamma_{22}^2 \cdot 0 = 0 - \frac{1}{2} 0 \]

\[ \Gamma_{22}^1 0 + \Gamma_{22}^2 (\phi'^2 + \psi'^2) = 2 \phi' \phi + \psi' \psi. \]

So

\[ \Gamma_{22}^1 = 0 \quad \Gamma_{22}^2 = \frac{\phi' \phi + \psi' \psi}{(\phi'^2 + \psi'^2)} \]

Now we would like to express e, f, g in terms of the \( \Pi_{ij} \). We can get started by noting

\[ (x u u)_v - (x u u)_u = 0. \]  \(1\)

\[ (x v u)_u - (x v u)_v = 0. \]  \(2\)

\[ (N u)_v - (N v)_u = 0. \]  \(3\)

Writing out \(1\) in the basis \( X_u, X_v, N \) we get

\[ A_1 X_u + B_1 X_v + C_1 N = 0. \]
for some $A_1, B_1, C_1$. Of course, $x_u, x_v, N$ are linearly independent, so this is a system of 3 equations

$$A_1 = 0, \quad B_1 = 0, \quad C_1 = 0.$$

Let's work these out. We see that

$$(x_{uu})_v = \left( \Pi^1_{ii} \right)_v x_u + \left( \Pi^4_{ii} \right)_v x_{uu}$$

$$+ \left( \Pi^2_{ii} \right)_v x_v + \left( \Pi^2_{ii} \right)_v x_{uv}$$

$$+ e_v N + e N_v$$

$$= \left( \Pi^1_{ii} \right)_v x_u + \left( \Pi^1_{ii} \Pi^1_{12} x_u + \Pi^1_{ii} \Pi^2_{12} x_v + \Pi^1_{ii} \Pi^0_{12} N \right)$$

$$\left( \Pi^2_{ii} \right)_v x_v + \left( \Pi^2_{ii} \Pi^1_{22} x_u + \Pi^2_{ii} \Pi^2_{22} x_v + \Pi^2_{ii} \Pi^0_{22} N \right)$$

$$+ e_v N + e \alpha_{12} x_u + e \alpha_{22} x_v$$

$$(x_{uv})_u = \left( \Pi^4_{12} \right)_u x_{uu} + \left( \Pi^4_{12} \right)_u x_{uv}$$

$$+ \left( \Pi^2_{12} \right)_u x_v + \left( \Pi^2_{12} \right)_u x_{uv}$$

$$+ f_u N + f N_u$$
\[
\begin{align*}
&= (\Gamma^1_{12}) \chi \chi + \Gamma^1_{12} \Gamma^1_{11} \chi \chi + \Gamma^1_{12} \Gamma^2_{11} \chi \chi + \Gamma^1_{12} \chi \chi + \Gamma^1_{12} \chi \chi + \\
&+ \Gamma^2_{12} \chi \chi + \Gamma^2_{12} \Gamma^1_{21} \chi \chi + \Gamma^2_{12} \Gamma^2_{21} \chi \chi + \Gamma^2_{12} \chi \chi \chi \chi \\
&+ \chi \chi \chi \chi + \chi \chi \chi \chi + \chi \chi \chi \chi
\end{align*}
\]

If we isolate \( B_2 \), the coefficient of \( \chi \chi \), in all this, we get

\[
\Gamma^1_{12} \Gamma^2_{11} \chi \chi + \Gamma^2_{12} \Gamma^2_{21} \chi \chi + \chi \chi \chi \chi = \text{e} a_{22} = \text{(Gauss formula)}
\]

This implies

\[
\text{e} a_{22} - \text{f} a_{21} = \text{a bunch of Christoffel symbols.}
\]

But by the Weingarten equations,

\[
\begin{align*}
a_{22} &= \frac{\text{f} E - \text{g} E}{\text{E} \text{G} - \text{F}^2} \\
a_{21} &= \frac{\text{e} F - \text{f} E}{\text{E} \text{G} - \text{F}^2}
\end{align*}
\]

so the lhs is

\[
\frac{\text{e} F \text{f} E - \text{e} \text{g} E - \text{f} \text{e} F + \text{f}^2 E}{\text{E} \text{G} - \text{F}^2} = -E \frac{\text{e} \text{g} \text{f}^2}{\text{E} \text{G} - \text{F}^2} = -E K.
\]
We have proved that
Theorem. Gauss curvature is invariant under isometries.

Pause to celebrate! Gauss called this the "Theorema Egregium" because it was so dam hard to prove.

Consequence: The helicoid and catenoid have the same Gauss curvature at corresponding points. (!)

(This is not obvious – why should they have the same product of principal curvatures when we can’t see why they would even have the same p.c.’s to start?)
Wrapping up, we observe that repeating this procedure for $A_1$, the coefficient of $Xu$ gives us an expression for $FK$.

For the coefficient of $N$, called $C_1$, we get

$$\left( \Pi_1^1 g + e \right) f + \Pi_1^2 g + e_v = \Pi_2^1 e + \Pi_2^2 f + f_u$$

or

$$e_v - f_u = e \Pi_1^1 + f \left( \Pi_2^2 - \Pi_1^1 \right) - g \Pi_1^2. \quad (*)$$

If we repeat the process for $C_2$, we get another expression for $FK$, an expression for $GK$ and

$$f_v - g_u = e \Pi_2^4 + f \left( \Pi_2^2 - \Pi_1^1 \right) - g \Pi_2^2. \quad (\star)$$

These equations $(\star)$ are called Mainardi–Codazzi equations.
Now recall that the $\Pi^k_{ij}$ are really expressed in terms of $E, F, G$, so the Gauss eqns and the M-C equations are really eqns relating $E, F, G$ and $e, f, g$.

Are these the only such relations?

Yes. (Bonnet's theorem).