Differential Geometry - Lecture 2

Last time we ended by defining a regular curve

$$\alpha(s) : (a, b) \rightarrow \mathbb{R}^3$$

so that $$\alpha'(s) \neq \mathbf{0}$$. 

And the curvature $$|\alpha''(s)| = \kappa(s)$$ and normal vector $$\hat{n}(s)$$ of a curve

$$\alpha''(s) = \kappa(s) \hat{n}(s).$$

Observe that $$\hat{n}(s) \cdot \alpha'(s) = 0$$:

$$\frac{d}{ds} (\alpha'(s) \cdot \alpha'(s)) = \frac{d}{ds} 1 = 0.$$ 

$$2 \alpha''(s) \cdot \alpha'(s) = 2 \kappa(s) (\hat{n}(s) \cdot \alpha'(s)).$$
Definition. The plane determined by $\alpha'(s)$ and $\hat{n}(s)$ is called the osculating plane at $s$.

We will now restrict our attention to curves where $\alpha''(s)$ does not vanish, so this plane is well-defined.

We denote $\alpha'(s)$ by $\hat{T}(s)$. Then we define the binormal $\hat{b}(s)$ by

$$\hat{b}(s) = \hat{T}(s) \times \hat{n}(s).$$

The binormal measures the rate at which $\alpha(s)$ is leaving the osculating plane.
Let's compute $\mathbf{b}'(s)$:

$$
\mathbf{b}'(s) = \frac{d}{ds} \mathbf{T}(s) \times \mathbf{n}(s)
= \mathbf{T}'(s) \times \mathbf{n}(s) + \mathbf{T}(s) \times \mathbf{n}'(s)
= \mathbf{T}(s) \times \mathbf{n}'(s).
$$

In particular,

$$
\mathbf{b}'(s) \cdot \mathbf{b}(s) = 0 \quad \text{(since $\mathbf{b}(s)$ is unit)}
$$

$$
\mathbf{b}'(s) \cdot \mathbf{T}(s) = 0 \quad \text{(by above)}
$$

so $\mathbf{b}'(s)$ must be a scalar multiple of $\mathbf{n}(s)$: we define $\gamma(s)$ by

$$
\mathbf{b}'(s) = \gamma(s) \mathbf{n}(s).
$$

Definition. The number $\gamma(s)$ is called the torsion of $\alpha$ at $s$.

We note that plane curves have zero torsion, and that any curve with nonvanishing curvature and vanishing torsion is a plane curve.
We call \( \hat{t}(s), \hat{n}(s), \hat{b}(s) \) the 
\textbf{Frenet frame} on \( \alpha(s) \). We 
\textbf{know}

\[
t'(s) = \chi(s) \hat{n}(s)
\]

\[
n'(s) = \frac{d}{ds} (\hat{b}(s) \times \hat{t}(s))
= \hat{b}'(s) \times \hat{t}(s) + \hat{b}(s) \times t'(s)
= \gamma(s) \hat{n}(s) \times \hat{t}(s) + \hat{b}(s) \times \chi(s) \hat{n}(s)
= -\gamma(s) \hat{b}(s) - \chi(s) \hat{t}(s).
\]

\[
b'(s) = \gamma(s) \hat{n}(s)
\]

\textbf{These are the Frenet formulas.} 
\textit{(We'll use these later.)}

\textbf{It seems like bending} \( (\chi(s)) \) \textbf{and}
\textbf{twisting} \( (\gamma(s)) \) \textbf{encapsulate all the}
\textbf{possible deformations of a space}
\textbf{curve.}

\textbf{This fact is expressed by}
Fundamental Theorem of Local Theory of Curves: Given differentiable functions \( K(s) > 0 \) and \( T(s) \) on \( I \), there exists a regular parametrized curve \( \alpha: I \to \mathbb{R}^3 \) so that

- \( S \) is the arclength of \( \alpha \)
- \( K(s) \) is the curvature of \( \alpha \)
- \( T(s) \) is the torsion of \( \alpha \)

Further, any other curve \( \tilde{\alpha}(s) \) with the same curvature and torsion differs from \( \alpha(s) \) by a rigid motion.

Another way of saying this; usually we use three functions \( x(s), y(s), z(s) \) to specify a curve. If \( \alpha(s) \) is parametrized by arclength, then

\[
x'^2(s) + y'^2(s) + z'^2(s) = 1
\]

so two "should" suffice (we could integrate up from \( z'(s) \) to \( z(s) \)).
We won't prove existence (that's the theory of ODE's) but we will prove uniqueness.

Proof (of part 2).

Note that arclength, curvature, and torsion are all invariant under rigid motions.

So suppose \( \alpha(s) \) and \( \tilde{\alpha}(s) \) have the same \( \mathbf{x}(s) \) and \( \mathbf{y}(s) \); we can certainly arrange for

\[
\begin{align*}
\mathbf{T}(0) &= \mathbf{T}(0) \quad \alpha(0) &= \tilde{\alpha}(0) \\
\mathbf{n}(0) &= \tilde{\mathbf{n}}(0) \\
\mathbf{b}(0) &= \tilde{\mathbf{b}}(0)
\end{align*}
\]

by a rigid motion of \( \tilde{\alpha}(s) \).

Now let's consider a kind of "total distance" between the frames \( \mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s) \) and \( \mathbf{T}(s), \tilde{\mathbf{n}}(s), \tilde{\mathbf{b}}(s) \):
We take
\[ d(s) = |t(s) - T(s)|^2 + |n(s) - \overline{n}(s)|^2 + |b(s) - \overline{b}(s)|^2. \]

Then
\[ d'(s) = \frac{d}{ds} (t(s) - T(s)) \cdot (t(s) - T(s)) \]
\[ + \frac{d}{ds} (n(s) - \overline{n}(s)) \cdot (n(s) - \overline{n}(s)) \]
\[ + \frac{d}{ds} (b(s) - \overline{b}(s)) \cdot (b(s) - \overline{b}(s)) \]
\[ = 2 \left( t'(s) - \overline{t}'(s) \right) \cdot (t(s) - T(s)) \]
\[ + 2 \left( n'(s) - \overline{n}'(s) \right) \cdot (n(s) - \overline{n}(s)) \]
\[ + 2 \left( b'(s) - \overline{b}'(s) \right) \cdot (b(s) - \overline{b}(s)) \]

Now using the Frenet equations:
\[ = 2 x(s) \left( n(s) - \overline{n}(s) \right) \cdot (t(s) - T(s)) \]
\[ - 2 y(s) \left( t(s) - T(s) \right) \cdot (n(s) - \overline{n}(s)) \]
\[ - 2 z(s) \left( b(s) - \overline{b}(s) \right) \cdot (n(s) - \overline{n}(s)) \]
\[ + 2 z(s) \left( n(s) - \overline{n}(s) \right) \cdot (b(s) - \overline{b}(s)) \]
\[ = 0. \]
Thus \( d'(s) = 0 \) and so, since \( d(0) = 0, \ d(s) \equiv 0. \) This means that \( t(s), n(s), b(s) = \bar{t}(s), \bar{n}(s), \bar{b}(s) \) everywhere.

In particular, since

\[
\alpha'(s) = t(s) = \bar{t}(s) = \bar{\alpha}'(s)
\]

we have

\[
\frac{d}{ds} (\alpha(s) - \bar{x}(s)) = 0.
\]

But this means

\[
\alpha(s) = \bar{\alpha}(s) + \bar{z}
\]

for some constant vector \( \bar{z}, \) and since \( \alpha(0) = \bar{x}(s) \) by assumption, we must have \( \alpha = \bar{x}, \) completing the proof.
Notice that we used the Frenet frame (and the assumption that $K(s) \neq 0$) in a really non-trivial way: if we relaxed that assumption we’d be faced with examples like:

(And our theorem would be false!)

$\Rightarrow$ Prove $\kappa \equiv 0 \iff$ planar, $\kappa \equiv 0 \iff$ linear

Second remark. We observe that given any regular curve $\alpha : I \to \mathbb{R}^3$ we can find $B : J \to \mathbb{R}^3$ parametrized by arclength with the same trace as $\alpha$. 
If we let
\[ s(t) = \int_{t_0}^{t} |x'(t)| \, dt \]
then since \( s'(t) = |x'(t)| \neq 0 \), this function has a differentiable inverse "\( \mathcal{F}(s) \)" by the inverse function theorem.

Further, since "\( \mathcal{F}(s(t)) = t \)", we have "\( \mathcal{F}'(s(t)) \). \( ds'(t) = 1 \).

So
\[
\frac{d}{ds} \alpha(\mathcal{F}(s)) = \alpha'(\mathcal{F}(s)) \cdot \mathcal{F}'(s)
\]
and
\[
\frac{d}{ds} \alpha(\mathcal{F}(s)) = |\alpha'(\mathcal{F}(s))| \cdot \mathcal{F}'(s)
= s'(\mathcal{F}) \cdot \mathcal{F}'(s)
= 1.
\]
Thus $\alpha(t(s)) = \beta(s)$ is an arclength parametrized curve with the same trace as $\alpha$.

We now define:

**Definition.** If $\alpha(t)$ is any regular parametrized curve (that is, $\|\alpha'(t)\| \neq 0$) we define **curvature** and **torsion** for $\alpha(t)$ at $t$ by saying

$$k(t) = \text{the curvature of the reparametrization of } \alpha \text{ by arclength at } t$$

$$\tau(t) = \text{the torsion of the reparametrization of } \alpha \text{ by arclength at } t.$$ 

**Note:** The fundamental theorem still works!