The Four-Vertex Theorem and beyond!

We begin by introducing a key idea in the study of curves.

Definition. The tangent indicatrix of a parametrized differentiable curve \( \alpha(s) \) is the curve \( t(s) = \alpha'(s) \).

Note that the tangent indicatrix, or \textit{tanrix}, is a curve on the unit sphere.

For a plane curve, we can be more specific: \( t(s) \) lies on \( S^2 \).

Even though \( \alpha(s) \) is unit speed, \( t(s) \) probably is not. In fact,
Proposition. The velocity $|\hat{\varepsilon}'(s)|$ of the tangent is the curvature $\kappa(s)$ of $\alpha$ at $s$.

Proof. This is pretty much the definition of curvature.

For a plane curve, we can write down the length of the tangent as the total curvature of the curve $\alpha$:

$$\text{total curvature} = \int_0^\ell \kappa(s) \, ds = \int_0^\ell |\hat{\varepsilon}'(s)| \, ds$$

$$= \text{length } \hat{\varepsilon}(s).$$

For a plane curve, we can be more specific; $\hat{\varepsilon}(s)$ lies on the unit circle, and we know...
Proposition. The total curvature of a closed plane curve is a multiple of $2\pi$.

Proof. The tangent of a must also be a closed curve on the unit circle; all such curves have length a multiple of $2\pi$.

We can describe the position of $\mathbf{t}(s)$ on the unit circle by some angle $\Theta(s)$.

![Diagram showing $\mathbf{t}(s)$ and $\Theta(s)$]

We define:

\[ I = \frac{1}{2\pi} \int \Theta'(s) \, ds \]

Definition. The rotation index of a plane curve $\mathbf{a}(s)$ is given by

\[ I = \frac{1}{2\pi} \int \Theta'(s) \, ds \]
While \( \Theta(s) \) is defined only up to a multiple of \( 2\pi \) by the above, we can certainly define \( \Theta(0) \) to be between 0 and \( 2\pi \), and then extend this definition to all \( \Theta(s) \) by observing:

a) \( \Theta'(s) \) is well-defined

b) \( \Theta(s) = \int_0^s \Theta'(x) \, dx + \Theta(0) \) defines \( \Theta(s) \) uniquely.

With these observations, we can write the definition. The rotation index of a plane curve \( \alpha(s) \) is given by

\[
I = \frac{1}{2\pi} (\Theta(l) - \Theta(0))
\]

where \( l \) is the length of \( \alpha \).
Notice that

Lemma. If \( \alpha(s) \) is a closed plane curve, its rotation index \( I \) is an integer.

Proof. \( \hat{\tau}(s) \) must also be closed, thus \( \Theta(0) \) and \( \Theta(2) \) differ by a multiple of \( 2\pi \).

Examples.

\begin{align*}
\text{I} &= -1 \\
\text{I} &= +1
\end{align*}
Theorem. A simple closed plane curve (one with no self-intersections) has rotation index $\pm 1$.

Equivalently (though we won't prove it)

The proof will come later, but is prefigured by the last example; roughly speaking, it costs as much angle to come out of a "pocket" as one gains going in.
We now introduce just a little more terminology!

**Definition.** A plane curve is convex if it lies on one side of each of its tangent lines.

![Convex and non-convex curves](image)

Equivalently (though we won't prove it) a curve is convex if any two points on the curve can be connected by a line inside the curve.

![Convex and non-convex curves](image)
Definition. A vertex of a plane curve \( \alpha(s) \) is a point where \( K'(s) = 0 \) (a critical point of curvature).

Examples

\[ \begin{array}{c}
\text{vertices}
\end{array} \]

Proof. By construction,

Thus

Note that this answers our challenge problem from the first class.

We now start on an amazing theorem:

Theorem: A simple closed convex curve has at least four vertices.
Before starting, we need a lemma.

Lemma. If $\alpha(s) = (x(s), y(s))$, $\alpha'(s) = (\cos \theta(s), \sin \theta(s))$, and $A, B, C$ are real numbers, then

$$\int_0^l (Ax(s) + By(s) + C) \theta''(s) \, ds = 0.\tag{as usual, $l$ is the length of the curve $\alpha$)$$

Proof. Observe that

$$x'(s) = \cos \theta(s) \quad y'(s) = \sin \theta(s)$$

So

$$x''(s) = -\sin \theta(s) \theta'(s) \quad y''(s) = \cos \theta(s) \theta'(s)$$

$$= -y'(s) \theta'(s) = x'(s) \theta'(s).$$

Now we can play a game! Since $x''(s)$ and $y''(s)$ are periodic functions of $s$, and $x'(s)$, $y'(s)$ are as well,

$$\int_0^l x''(s) \, ds \quad \int_0^l y''(s) \, ds = 0.$$
But we know
\[ \int_0^l a(s)x'(s)\,ds = \int_0^l -y(s)\theta'(s)\,ds \]

Let's do the next step in slow motion:
\[ \frac{d}{ds} y(s)\theta'(s) = y'(s)\theta'(s) + y(s)\theta''(s) \]

Integrating both sides from 0 to \( l \), we see

\[ y(l)\theta'(l) - y(0)\theta'(0) = \int_0^l y'(s)\theta'(s)\,ds + \int_0^l y(s)\theta''(s)\,ds \]

Now both \( y(s) \) and \( \theta'(s) \) are periodic since \( \alpha \) is a closed curve! Thus

\[ -\int_0^l y'(s)\theta'(s)\,ds = \int_0^l y(s)\theta''(s)\,ds \]

Thus

\[ \int_0^l y(s)\theta''(s)\,ds = 0. \]
We can play exactly the same game with the other term
\[ 0 = \int_0^l y''(s) \, ds = \int_0^l x'(s) \theta'(s) \, ds \]
\[ = -\int_0^l x(s) \theta''(s) \, ds. \]
Now we (last) observe that
\[ \int_0^l \theta''(s) \, ds = \theta'(l) - \theta'(0) \]
\[ = 0. \]
But these three integrals are the components of our original integral!
\[ \int_0^l (A x(s) + B y(s) + C) \theta''(s) \, ds \]
\[ = A \int_0^l x(s) \theta''(s) \, ds + \]
\[ B \int_0^l y(s) \theta''(s) \, ds + \]
\[ C \int_0^l \theta''(s) \, ds. = 0. \]
This proves the lemma.
We now prove the theorem!

Proof. Let \( \dot{\alpha}(s) \) be parametrized by arclength on \([0, \ell]\). Since \( K(s) \) is continuous, it reaches a maximum and minimum value on \([0, \ell]\).

These points are vertices of \( \alpha \), at

\[ \dot{\alpha}(s_1) = \vec{p} \quad \text{and} \quad \dot{\alpha}(s_2) = \vec{q}. \]

Connect these with a line \( L \), and let \( \beta \) and \( \gamma \) be the arcs of \( \alpha \) on either side of \( L \), determined by these points.
Now we claim $\beta$ and $\gamma$ lie on opposite sides of $L$; this follows from the convexity of $\alpha$.

We now want to play another game: we observe that $\Theta''(s) = K'(s)$. Further, at a vertex of $\alpha$, $K'(s) = \Theta''(s) = 0$.

Now if there are no other vertices on $\alpha$, $K'(s)$ must have one sign on $\beta$ and the other sign on $\gamma$.

Further, if $Ax + By + C < 0$ is the equation of $L$, then

So we have (choosing the sign of $A, B, C$ if we have to)
the estimate

\[(A x(s) + B y(s) + C) \geq K'(s) > 0\]

on both \(B\) and \(y\). But then

\[\int_0^1 (A x(s) + B y(s) + C) K'(s) > 0.\]

But \(K'(s) = \Theta''(s)\), and we just showed

\[\int_0^1 (A x(s) + B y(s) + C) \Theta''(s) = 0.\]

Thus the function \(K'(s)\) must change sign on \(B\) or \(y\), creating a third vertex! Suppose this happens on \(y\); we have just shown that the function \(K^*(s)\) has a local max or min on \([s_1, s_2]\).

But \(s_1\) is the global max and \(s_2\) the global min of \(K(s)\), so \(K'(s)\) must have the same sign near \(s_1\) and \(s_2\) - negative.

Drawing the plot of maxes and mins from 2200, we see
third vertex

\[ s_1 \rightarrow \cdot \cdot \cdot + \rightarrow - s_2 \]

there must be a fourth vertex in here!

This completes the proof!

The theorem is still true for simple (but non-convex) curves, but the proof is harder!

Is there a converse? \( \text{Yes!} \)

The rotation index is too much fun to pass up.