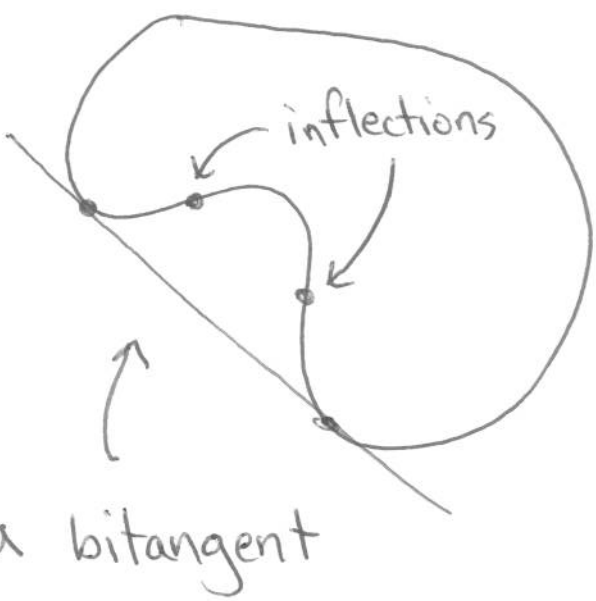
The Fabricius-Bjerre Theorem.

Let's consider plane curves a bit more. We've learned that a closed planar curve has at least 4 vertices (if convex). What about nonconvex curves?



a bitangent

2 bitangents

4 bitangents, 2 crossings.

An amazing fact is that there's a connection between

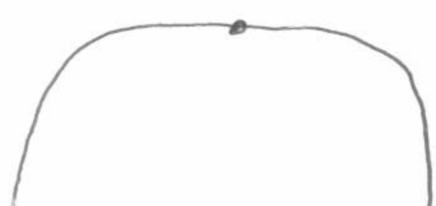
Definition. A <u>bitangent</u> is a line which is tangent to a plane curve in two places.

positive bitangent

negative bitangent

Definition. An <u>inflection point</u> is a point where the signed curvature of a plane curve changes sign.

inflection



<u>not</u> an inflection

Definition. A <u>double point</u> is a point where the curve crosses itself (and the tangent vectors are linearly independent).

double point

not a double point

If we count

T_+ = # pos. bitangents

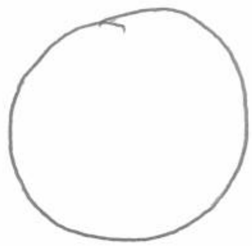
T_- = # neg. bitangents

I = # inflections

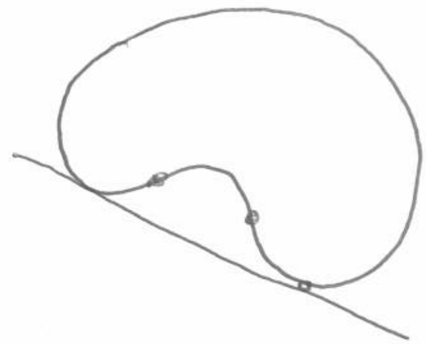
D = # double points

there's an amazing relationship!

Let's try to guess it.

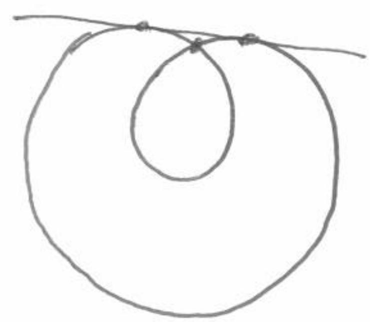


no inflections,
no double pts,
no bitangents
$\Downarrow$
$$aI + bT_+ + cT_- + dD = 0$$

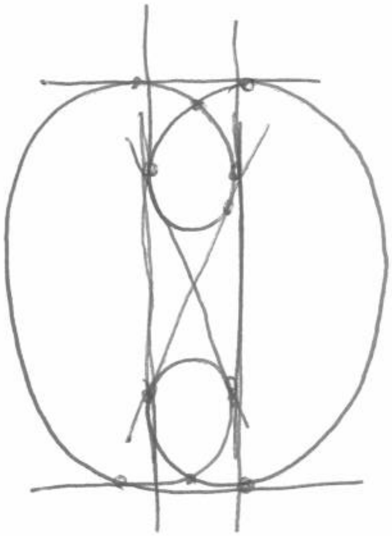


$T_+ = 1,\ I = 2,\ T_- = 0,\ D = 0$
so
$$2a + 1b = 0$$

$T_+ = 2,\ D = 1,\ I = 2$
so
$$2a + 2b + d = 0$$



$T_+ = 1,\ D = 1,\ T_- = 0,\ I = 0$
so
$$1b + 1d = 0$$



$T_+ = \cancel{2}\ 4,\ T_- = 2,\ D = 2,\ I = 0$

so

$$4\cancel{2}b + 2c + 2D = 0.$$

We now have four equations in four unknowns;

$$
\begin{aligned}
2a + 1b &&= 0 \quad &(1)\\
2a + 2b &+1d &= 0 \quad &(2)\\
1b &+1d &= 0 \quad &(3)\\
4\cancel{2}b + 2c &+2d &= 0 \quad &(4)
\end{aligned}
$$

We see $b = -d$ (3), $a = -\frac{1}{2}b$ (2), and that (2) is a consequence of (1) and (3). (rats).
Also $c = d$ by (4).

We are left with (set $b=1$)

$$-\frac{1}{2}I + T_+ - T_- - D = 0$$

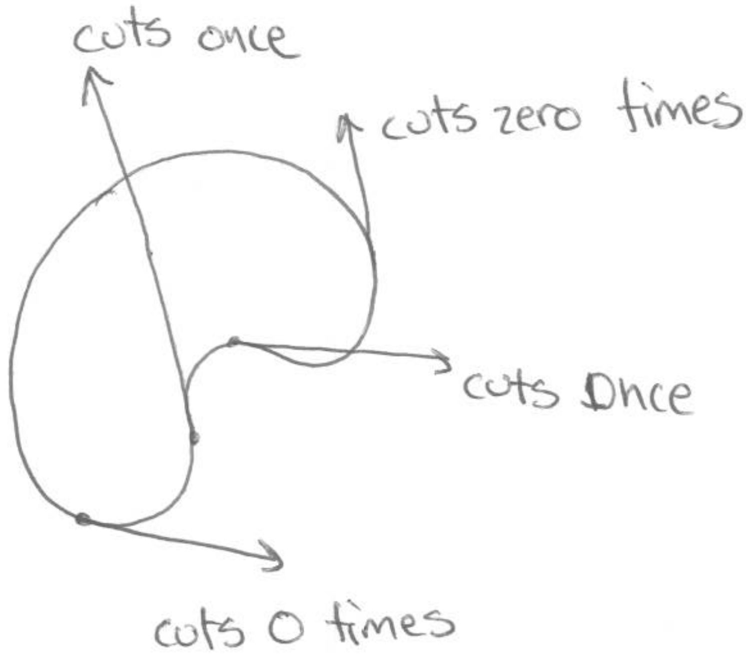
or any multiple of this equation.

Theorem (Fabricius-Bjerre, 1962).

$$-\frac{1}{2}I + T_+ - T_- - D = 0$$

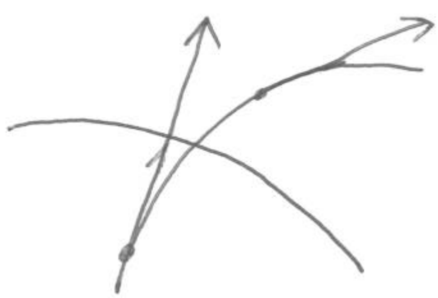
for any smooth closed plane curve.

Proof.



Proof. Orient the curve and consider the number of times the positive ray in the tangent direction cuts the curve, calle this $N(s)$.

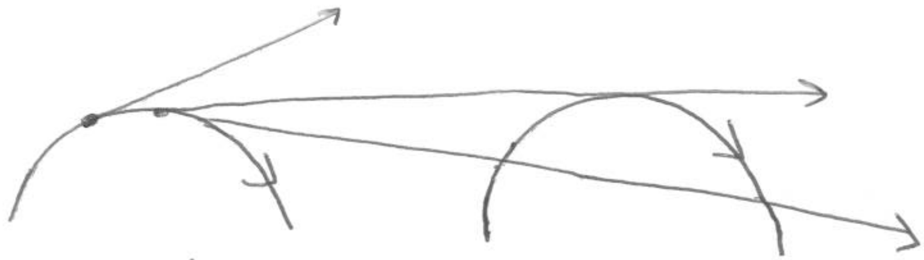
N is a periodic function with jumps. The sum of all the jumps is zero, since N is periodic.



passing a double point makes N drop by 1. This happens twice for every double point.

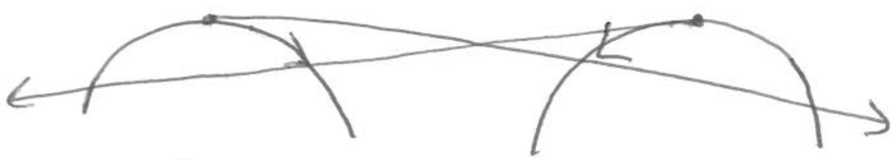
passing an inflection makes N drop by 1.

Passing a double tangent can cause several things to happen.



type a

N jumps by +2



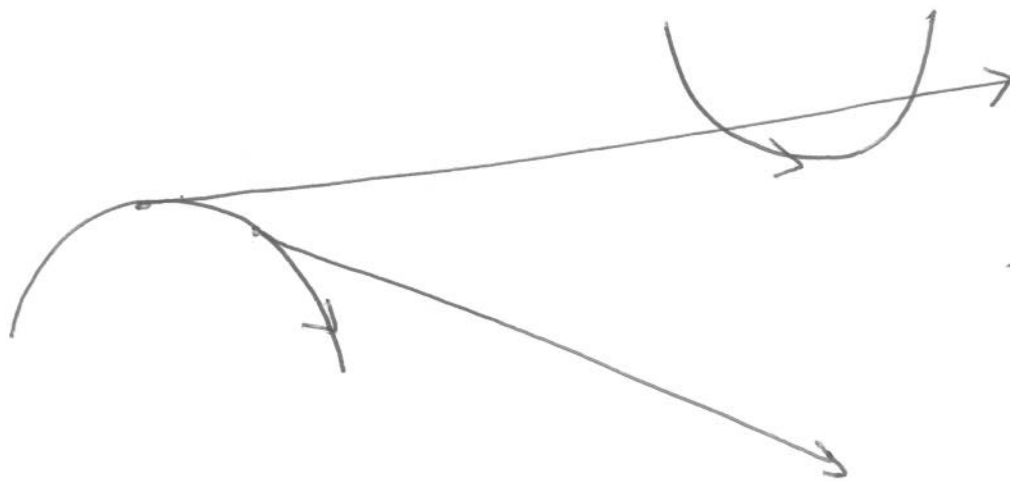
type b

N jumps by +4 (+2 on each side)

type c

N doesn't change.

Or for a - bitangent,



type i

N ~~ch~~ jumps by -2

type ii

N jumps by -4

type iii

N doesn't change

So we see

$$-2D - I + 2(T_+^a) + 4(T_+^b)$$

$$* - 2(T_-^i) - 4(T_-^{ii}) = \text{sum of jumps}$$
$$= 0.$$

Now if we reverse orientation on the curve,* to get a new curve $\tilde{\alpha}$,

$\tilde{D} = D$ (same # of double pts)

$\tilde{I} = I$ (same # of inflection pts)

$\tilde{T}_+^a = T_+^a$ (type a $\leftrightarrow$ type a)

$\tilde{T}_+^b = T_+^c$ (type b $\leftrightarrow$ type c)

$\tilde{T}_+^c = T_*^b$

$$\tilde{T}_-^{i} = T_-^{i} \qquad (\text{type } i \leftrightarrow \text{type } i)$$

$$\tilde{T}_-^{ii} = T_-^{iii}$$

$$\tilde{T}_-^{iii} = T_-^{ii} \qquad (\text{type } ii \leftrightarrow \text{type } iii)$$

So we have two equations:

$$-2D - I + 2(T_+^{a}) + 4(T_+^{b}) - 2(T_-^{i}) - 4(T_-^{ii}) = 0$$

$$-2D - I + 2(T_+^{a}) + 4(T_+^{c}) - 2(T_-^{i}) - 4(T_-^{iii}) = 0$$

or

$$-4D - 2I + 4(T_+^{a} + T_+^{b} + T_+^{c}) - 4(T_-^{i} + T_-^{ii} + T_-^{iii}) = 0,$$

which is

$$-\frac{1}{2}I + T_+ - T_- - D = 0$$

if we divide by 4 and rearrange terms. □