Stuff turning inside out!

The Whitney-Gávenstein Theorem.

Let's go back to the rotation index for a moment. We have a map from closed curves to $S^1$ given by the tangent indicatrix $\tilde{z}(s)$. The number of times $\tilde{z}(s)$ wraps around that circle is called the rotation index of $\alpha$.

We notice that $I$ counts "loops" in $\alpha$.

![Diagram](image.png)

$I = \pm 1$  
$I = \pm 2$

To what extent are these loops stable features of $\alpha$? Can we classify all curves by counting their "loop number"?
We first need a way to define equivalence of curves: we recall that

Definition. A curve $\alpha(t)$ is regular if $\alpha'(t) \neq 0$.

We then have:

Definition. A regular homotopy between two curves $\alpha(t)$ and $\beta(t)$ is a $C^1$ continuous map $H(t, x) : [a, b] \times [0, 1] \to \mathbb{R}^2$ so that $H(t, 0) = \alpha(t)$, $H(t, 1) = \beta(t)$ and $H(t, x_0)$ is a regular parametrized differentiable curve as a function of $t$.

We can think of $H$ as a movie showing $\alpha$ sliding over to $\beta$. 

\[ \begin{align*}
\text{at } x=0 & \quad \text{at } x=0.1 & \quad \text{at } x=0.5 & \quad \text{at } x=0.7 & \quad \text{at } x=1 \\
\end{align*} \]
Notice that some moves are forbidden by our definition of regular homotopy.

Since $H$ is $C^1$ (as a function of $x$ and $t$),

$$\frac{\partial H}{\partial t}(x, t)$$ is continuous in $x$.

So the tangent vectors of the intermediate curves $H(t, x)$ depend continuously on $x$.

So can we... pull a loop tight?

\[ \begin{array}{c}
\text{No! Can we make a corner?}
\end{array} \]

\[ \begin{array}{c}
\text{No! In particular,}
\end{array} \]

Proposition. The rotation index of $H(t, x)$ depends continuously on $x$. Thus it's constant.
Proof. The rotation index is defined as the total angle swept out by the tangent indicatrix of \( H(t,x) \).

Since the tangent indicatrix depends continuously on \( x \), this angle does too.

We now know: You can't turn a circle inside out!

\[
\begin{align*}
I = 0 & \quad \text{is not regularly homotopic to} \quad I = +1 \\
I = -1 & \quad \text{reg. homotopic to} \quad I = +2
\end{align*}
\]

Yes! Are all curves with the same rotation index regularly homotopic?

This is the Whitney-Gravenstein theorem!
Now what about surfaces?

At every point on a surface, there is a normal vector \( \hat{n}(x) \).

These normal vectors define a Gauss map from \( S \) to the unit sphere

\[ g: S \to S^2 \]

If \( S \) is a (funny shaped) sphere, this is (topologically) a map

\[ g: S^2 \to S^2 \]

with a "rotation index" or *degree*, measuring how many times \( g \) wraps around \( S^2 \).

Now if we turn the sphere inside out, we compose

\[
S^2 \xrightarrow{\hat{n} \cdot \mathbf{x} - \mathbf{x}} S^2 \xrightarrow{g} S^2 \xrightarrow{\mathbf{x} \mapsto -\mathbf{x}} S^2
\]

position vectors reversed, normal vectors reversed, too!
It turns out that degree multiplies when you compose maps, and the degree of the antipodal map is \(-1\).

So the inside-out sphere has the same "rotation index" as the original sphere!

Is there a "Whitney-Gravenstein Theorem for surfaces" which would let us conclude these spheres are regularly homotopic?!!

Let’s find out...