Math 6250: The intrinsic gradient.

This extra material accompanies the lecture notes on "Surfaces and the First Fundamental Form". This material is assigned for students enrolled in MATH 6250, but is an extra-credit assignment for students in MATH 4250.

Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ has a directional derivative in any direction $\vec{v} \in \mathbb{R}^n$ given by the limit

$$D_{\vec{v}} f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{v}) - f(\vec{x})}{h}.$$

As we studied in our notes on "The Gradient and Hessian", if we define the gradient of f at $\vec{x}$ to be the special vector $\nabla f(\vec{x}) = (\partial f/\partial x_1(\vec{x}), \dots, \partial f/\partial x_n(\vec{x}))$ we have $D_{\vec{v}} f(\vec{x}) = \langle \vec{v}, \nabla f \rangle$.

If the function $f(\vec{x})$ is defined on a curved surface $S \subset \mathbb{R}^3$, we still want to be able to understand what it means to take directional derivatives of the function. We may assume[a] that $f\colon S \to \mathbb{R}$ is the restriction of a function $f\colon \mathbb{R}^3 \to \mathbb{R}$.

Definition. *The directional derivative $D_{\vec{v}} f(\vec{p})$ (in $\mathbb{R}^3$) at p is a linear function of directions $\vec{v} \in \mathbb{R}^3$. The restriction of $D_{\vec{v}} f(\vec{p})$ to vectors $\vec{v} \in T_pS$ is a linear function of directions $\vec{v} \in T_pS$. We can always[b] write this linear function as*

$$D_{\vec{v}} f(\vec{p}) = \langle \vec{v}, \nabla f(\vec{p}) \rangle_{I_p}.$$

for some vector $\nabla f(\vec{p}) \in T_pS$. We define this vector to be the intrinsic gradient *of f.*

What is the formula for ∇f? It is important to know it. But it is clearly not as simple as it used to be for functions defined on R^n. Finding a formula for ∇f will require us to understand the first fundamental form in some detail, using the theory we've developed. This leads us to an answer to our question "What is differential geometry for?":

Differential geometry tells you how to do calculus on a curved surface.

Let's begin. The *extrinsic gradient* of a differentiable function $f\colon S \to \mathbb{R}$ is a differentiable map $\nabla f\colon S \to \mathbb{R}^3$ which assigns to each point of S a vector $\nabla f(p) \in \mathbb{R}^3$ so that

$$Df_p(\vec{v}) = \langle \nabla f, \vec{v} \rangle_{\mathbb{R}^3}.$$

It's worth noting that this is the $\mathbb{R}^3$ inner product because we wrote the gradient as a 3-vector in space. Because we're referring to the "external" space $\mathbb{R}^3$, we call this gradient "extrinsic".

[a] In a more advanced course, we prove that if f is a differentiable function defined on a regular surface $S \subset \mathbb{R}^3$, we may always extend f to a differentiable function on a neighborhood of S. Further, although this extension is not unique, the directional derivatives in tangent directions don't depend on which extension we choose.

[b] When you take an advanced course in linear algebra, you'll learn that every finite-dimensional vector space V is isomorphic to its *dual space* V^* of linear maps $A\colon V \to \mathbb{R}$. Further, if V has an inner product $\langle -, - \rangle$, every $A\colon V \to \mathbb{R}$ in V^* may be written as $A(\vec{w}) = \langle \vec{v}, \vec{w} \rangle$ for a unique $\vec{v} \in V$.

1. (10 points) If E, F, G are the coefficients of the first fundamental form on S, then *as a vector in* $\mathbb{R}^3$ the extrinsic gradient is

$$\nabla f = \frac{f_u G - f_v F}{EG - F^2}\vec{x}_u + \frac{f_v E - f_u F}{EG - F^2}\vec{x}_v.$$

and hence the *intrinsic* gradient of f at p is

$$\nabla f = \left(\frac{f_u G - f_v F}{EG - F^2}, \frac{f_v E - f_u F}{EG - F^2}\right).$$

2. (10 points) Suppose S is the x-y plane with the parametrization $X(u, v) = (u, v, 0)$. Compute the coefficients of the first fundamental form and use the formula above to show that the intrinsic gradient $\nabla f = (f_u, f_v)$.

3. (10 points) We proved in the "Gradient and Hessian" homework that the gradient vector points in the direction of largest directional derivative. Let's prove that the corresponding property holds for the intrinsic gradient.

   Fix a p in S and consider the unit circle $|\vec{v}| = 1$ in $T_p(S)$. Prove that on this circle $Df_p(\vec{v})$ is maximized $\iff$ $v = \nabla f / |\nabla f|$.

   Hint: Parametrize the unit circle by $\vec{v} = (\cos\theta)\vec{x}_u + (\sin\theta)\vec{x}_v$ and differentiate w.r.t. θ.

4. (10 points) Consider a *level curve* $C = \{p \in S : f(p) = c\}$ on S, and suppose that the tangent vector to C at p is given by $T_C(p)$. Prove that $\langle \nabla f(p), T_C(p) \rangle_{I_p} = 0$ when $p \in C$; that is, that ∇f is normal to C everywhere, just as it is for the ordinary gradient of a function on $\mathbb{R}^n$.