Geodesics on $G$ and $S$ manifolds (cont)

We now consider more about the geodesics of Grassmann and Stiefel manifolds. We know that each

$$Y(t) = Q e^{X t} \text{In}_{p, K}$$

where $Q \in \text{SO}(n)$, $X = \begin{bmatrix} A - B^T \end{bmatrix}$, $\text{In}_{p, K} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ is a Stiefel geodesic inside $\text{Mat} \times \mathbb{R}$. But we would like a formula in terms of $Y(0) = y$ and $Y'(0) = H$.

We can compute

$$Y'(t) = \frac{d}{dt} Q e^{X t} X \text{In}_{p, K}$$

so

$$Y'(0) = Q X \text{In}_{p, K}$$

or

$$H = \left[ Y - \right] \begin{bmatrix} A - B^T \\ B & 0 \end{bmatrix} \text{In}_{p, K}$$
Multiplying by $Q^T$ on the left, we see

\[
\begin{bmatrix}
\leftarrow y^T \\
\uparrow \\
\uparrow \\
\uparrow \\
\downarrow \\
\downarrow \\
\ldots \\
\end{bmatrix} \begin{bmatrix}
\uparrow H \\
\downarrow \\
\end{bmatrix} = \begin{bmatrix}
A \\
B \\
\end{bmatrix}
\]

This lets us compute $Y^TH = A$.

We can now observe that since

\[
H^TH = -I_{n,k}^T [A - B^T] QT Q [A - B^T] I_{n,k}
\]

\[
= -I_{n,k}^T [AA - B^TB - ] I_{n,k}
\]

\[
= \mathbb{P} A^T A + B^TB,
\]

we have a way to solve for $B^TB$ as

\[
B^TB = H^TH - A^TA = H^TH - H^TYY^TH
\]

\[
= H^T(I - YY^T)H.
\]
We then can define $C := H^T(I-YY^T)H$.  

Now we prove

Theorem. If $Y(t) = Qe^{t[A-BF]}In,p$ with $Y(0) = Y$ and $Y'(0) = H$, then

$$Y(t) = YM(t) + (I-YY^T)H \int_0^t M(t) \, dt$$

where $M(t)$ is the solution to the second order constant coefficient ODE

$$M''(t) - AM'(t) + CM = 0, \quad M(0) = I_p, \quad M'(0) = A$$

and $A = Y^TH$ is skew-symmetric, while $C = H^T(I-YY^T)H$ is non-negative definite.

Proof. We start by observing that $M(t) = In,k e^{t[A-BF]}In,k$. To prove this, we just differentiate:
We'll need a little improvement to a previous lemma:

Lemma. \( \frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA}A. \)

Now we can check our matrices \( M(t) \) satisfy the claimed ODE.

We compute

\[
M(t) = I_k \quad I_{n,k}^T e^{t \begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}} I_{n,k}
\]

\[
M'(t) = A \quad I_{n,k}^T e^{t \begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}} I_{n,k}
\]

\[
M''(t) = (-A^T A - B^T B) \quad I_{n,k}^T e^{t \begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}} I_{n,k}.
\]

Thus, if we let \( I_{n,k}^T e^{t \begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}} I_{n,k} = M(t) \) again, we see that

\[
M''(t) - AM'(t) + CM(t)
\]

\[
= \left[ (-A^T A - B^T B) - AA + C \right] M(t)
\]

\[
= \left[ -B^T B + C \right] M(t) = 0, \quad \text{as claimed.}
\]
Farther, plugging in $t=0$ yields

$$M(0) = I_{k}, \quad M'(0) = A.$$  

By uniqueness of solutions to ODEs, we now know that

$$M(t) = I_{n,k} \, e^{t \begin{bmatrix} A & -B^T \end{bmatrix}} \, I_{n,k},$$

as desired.

We now consider some geometry. Suppose that

$$y = \begin{bmatrix} Y \end{bmatrix}_{n \times k}$$

is an orthonormal basis for $k$-dim subspace $S$.

Then for any $n \times k$ matrix $H$, $y^T H = \begin{bmatrix} \varepsilon \end{bmatrix}_{k \times k}$

the expression of $\Pi_S H$ in the column coordinates of $y$. 
Now we introduce a neat trick. Suppose we have an $n \times k$ "orthogonal" matrix $Y$, so that $Y^T Y = I_k$. Consider the $n \times n$ matrix $YY^T$. This is called the projector onto the column space of $Y$, since it orthogonally projects vectors in $\mathbb{R}^n$ onto this space.

To see this, note that for an $n \times 1$ vector $\hat{v}$,

$$Y^T \hat{v} = \text{dot products of } \hat{v} \text{ w/ cols of } Y$$

$$YY^T \hat{v} = \text{linear combo. of those cols. according to dot products.}$$

Of course, this means that

$$I - YY^T = \text{projector onto } (\text{col space } Y)^\perp.$$
These two projectors also have the following useful properties:

\[(YY^T)^2 = YY^T\]
\[(I - YY^T)^2 = (I - YY^T)\]
\[YY^T (I - YY^T) = (I - YY^T) YY^T = 0.\]

We now turn back to verifying

\[Y(t) = YM(t) + (I - YY^T) H \int_0^t M(t) dt.\]

Multiplying on the left by \(YY^T\), we see that we must show

\[YY^T Y(t) = YY^T YM(t)\]
\[= YM(t)\]

Now we know \(Y(t) = Q e^{tX} I_{n,k}\) and \(M(t) = I_{n,k} e^{tX} I_{n,k}\), so we can write this as

\[(YY^T Q) e^{tX} I_{n,k} = (YI_{n,k}^T) e^{tX} I_{n,k}\]
$$\begin{bmatrix} \uparrow & \vdots & \uparrow \\ \downarrow & \text{stuff} & \downarrow \\ {Y}_{k \times n} \end{bmatrix} = \begin{bmatrix} I_k & 0 \end{bmatrix} = \begin{bmatrix} I_n \end{bmatrix},$$
as desired.

Multiplying on the left by $I - YY^T$, we get to show

$$(I - YY^T) Y(t) = (I - YY^T) H \int_0^t M(t) \, dt$$

Now $Y(t) = Q \, e^{tX}$, where $Q = \begin{bmatrix} \vdots & \text{stuff orthogonal to } Y \end{bmatrix}$, so let's call the right-hand $n \times (n-K)$ block $Z$. We see that

$$(I - YY^T) \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ Z \end{bmatrix},$$
so the lhs obeys

$$(I - YY^T) Y(t) = \begin{bmatrix} 0 \\ Z \end{bmatrix} e^{tX} I_{n,k}.$$
Now we know
\[
(I - YY^T) Y(0) = (I - YY^T) Y = 0
\]
and
\[
(I - YY^T) H \int_0^t M(t) dt = 0.
\]
So we must only show
\[
\frac{d}{dt} (I - YY^T) Y(t) = \frac{d}{dt} (I - YY^T) H \int_0^t M(t) dt,
\]
or
\[
(I - YY^T) Y'(t) = (I - YY^T) H M(t) e.
\]
Now \(Y'(t) = QX e^{*X^T} t I_{n, k}\), so we want
\[
(I - YY^T) QX e^{*X^T} t I_{n, k} \neq (I - YY^T) H e^{*X^T} t I_{n, k}.
\]
It suffices to show
\[
(I - YY^T) QX = (I - YY^T) H I_{n, k}^T
\]
actually we need to show this since
\(e^x\) is orthogonal and \(I_{n, k}\) weakly orthogonal.
Now
\[ y'(0) = Q X I_{n,k} = H \]
so we can substitute this in on the right, leaving us to prove
\[(I - yy^T) Q X = (I - yy^T) Q X I_{n,k} \]
Now the right hand operator \( I_{n,k} I_{n,k}^T \)
is projection onto the first \( K \) columns.
Further, we previously saw
\[(I - yy^T) Q = \begin{bmatrix} 0 & Z \\ K & n-K \end{bmatrix} \]
Multiplying by \( X = \begin{bmatrix} A - B^T \\ B & 0 \end{bmatrix}_K \)
we get
\[(I - yy^T) Q X = \begin{bmatrix} 0 & Z \\ K & n-K \end{bmatrix} \begin{bmatrix} A - B^T \\ B & 0 \end{bmatrix}_K = \begin{bmatrix} ZB & 0 \end{bmatrix}_K \]
which is clearly invariant under projection to the first \( K \) columns. \( \square \)
(whew!).
That was lengthy (and, frankly, ugly), but in the end we got a prize.

Now how do we solve matrix ODES like

\[ M''(t) - AM'(t) + CM = 0. \]

The solution turns out to be given by solving the "quadratic eigenvalue problem"

\[ (\lambda^2 I - A\lambda + C)x = 0. \]

Such problems (for KxK matrices) generally have 2K eigenvalues and 2K eigenvectors. If \( \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_{2K} \end{bmatrix} \), then if \( X = \begin{bmatrix} X_1 & \cdots & X_{2K} \end{bmatrix}_{K \times 2K} \) the matrix of eigenvectors, and \( Z \) is a 2KxK matrix chosen so that \( XZ = I \), \( XLZ = A \), then

\[ M(t) = Xe^{A_t}Z. \]
We can check this directly:

\[ \text{M}'(t) = X \Lambda \, e^{\Lambda t} \, Z \]
\[ \text{M}''(t) = X \Lambda^2 \, e^{\Lambda t} \, Z \]

So

\[ \text{M}''(t) - AM'(t) + CM(t) = \]
\[ (X \Lambda^2 - A \Lambda \Lambda + C \Lambda) \, e^{\Lambda t} \, Z \]

But column-by-column, \( X \) must satisfy this matrix equation by construction.