Infinite-dimensional Grassmannians and Stiefel Manifolds. Curves.

We begin with three function spaces:

$$V_{open} = \{ C^\infty \text{ mappings } f : [0, 2\pi] \to \mathbb{R} \}$$

$$V_{even} = \{ f \in V_{open} \mid f(0) = f(2\pi) \}$$

$$V_{odd} = \{ f \in V_{open} \mid f(0) = -f(2\pi) \}.$$

On each space we take the L^2 inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) \cdot g(\theta)\, d\theta$$

~~with the~~

Definitions. The map

$$\Phi : (e, f) \to C(\theta) = (1/2) \int_0^\theta (e(x) + i f(x))^2 \, dx$$

is a map from V_{open}^2 to the space of plane curves.

We let

$$Z(e,f) = \{\theta \mid e(\theta) = f(\theta) = 0\}.$$

These are the points where the parametrization of the curve $c(\theta)$ is not regular (i.e. c is not an immersion of $[0,2\pi] \to \mathbb{C}$).

We define $S(V^2_{open})$ = the ~~unit~~ sphere in V^2_{open} so that $\|e\|^2 + \|f\|^2 = 2$. This sphere has a subset $S^0(V^2_{open})$ of pairs e, f so that $Z(e,f) = \emptyset$.

Last, we define the ~~sub~~manifold of immersions $Imm_{open} = Immersions([0,2\pi], \mathbb{C})$. The tangent space at a particular curve c is given by

$$T_c(Imm_{open}) = \{\text{vector fields } h : [0,2\pi] \to \mathbb{C}\}.$$

We can mod out by translation, noting that

$$T_c(\mathrm{Imm}_{open}/\mathrm{trans}) = T_c(\mathrm{Imm}_{open})/_{\text{constant fields}}.$$

There is a Riemannian metric on this (infinite dimensional) manifold given by

$$\langle h_1, h_2 \rangle_c = \frac{\quad}{2\ell(c)} \int_0^{2\pi} \frac{\langle h_1', h_2' \rangle \, d\theta}{|c'(\theta)|}$$

If we compose $c(\theta)$ with a scaling λ, we scale $h_1', h_2', |c'(\theta)|$, and $\ell(c)$ by λ so this metric is scale invariant.

Note that if θ is an arclength parametrization we have

$$\langle h_1(s), h_2(s) \rangle_c = \frac{1}{\ell(c)} \int_c \langle h_1'(s), h_2'(s) \rangle \, ds.$$

Theorem (Mumford-Michor-Shah, 2008).

The map

$$\Phi : S^{\circ}(V^2_{open}) \to \underbrace{\{ c \in Imm_{open} \mid \ell(c) = 1,\ c(0) = 0 \}}_{= \; Imm_{open} / \text{trans, scaling}}$$

is an isometric 2-fold covering, using the natural metric on $S^{\circ}(V^2_{open})$ and the given metric ~~Proof.~~ on Imm_{open}.

Proof.

We first show the map is $2 \to 1$ and surjective. Given any $c : [0, 2\pi] \to \mathbb{C}$, we can write

$$c'(\theta) = r(\theta) e^{i\psi(\theta)}$$

$$\sqrt{2c'(\theta)} = \sqrt{2r(\theta)}\, e^{i\psi(\theta)/2}$$

$$= \sqrt{2r(\theta)} \cos\left(\frac{\psi(\theta)}{2}\right) + i\sqrt{2r(\theta)} \sin\left(\frac{\psi(\theta)}{2}\right).$$

$$= e(\theta) + i f(\theta).$$

We see that

$$\left(\Phi(e,f)\right)(\theta) = \frac{1}{2}\int_0^\theta \left(e(t)+if(t)\right)^2 dt$$

$$= \frac{1}{2}\int_0^\theta 2c'(t)\,dt = c(\theta)-c(0) = c(\theta),$$

as desired. Since we can choose the sign of $\sqrt{2r(\theta)}$, ~~this~~ the map Φ is $2\to 1$.

We now show that the map is a (local) isometry. Suppose

$$\Phi(e,f) = c$$

and we have a tangent vector $(\delta e, \delta f)$ in the tangent space to $S^0(V^2_{open})$. The differential of Φ is given by

$$D\Phi_{(e,f)}(\delta e,\delta f) = \frac{1}{2}\int_0^\theta \left(e+\delta e + i(f+\delta f)\right)^2 dt$$

$$= \int_0^\theta (\delta e + i\delta f)(e+if)\,dt.$$

We can then compute that the corresponding variation of $c(\theta)$ is given by

$$h(\theta) = \int_0^{\theta} (\delta e + i\delta f)(e + if)\, d\cancel{\theta}t$$

This means that

$$h'(\theta) = \big(\delta e(\theta) + i\delta f(\theta)\big)\big(e(\theta) + if(\theta)\big).$$

To compute:

$$\langle h'(\theta), h'(\theta)\rangle_{\mathbb{R}^2} = h'(\theta)\overline{h'(\theta)}$$

$$= \big((\delta e(\theta))^2 + (\delta f(\theta))^2\big)\big(e(\theta)^2 + f(\theta)^2\big)$$

Now we see that since

$$c(\theta) = \frac{1}{2}\int_0^{\theta} \big(e(\theta) + if(\theta)\big)^2\, d\theta,$$

$$|c'(\theta)| = \frac{1}{2}\big(e(\theta)^2 + f(\theta)^2\big),$$

we have

$$\langle h(\theta), h(\theta)\rangle_c = \frac{1}{2\ell(c)} \int_0^{2\pi} \frac{(\delta e^2 + \delta f^2)(e^2+f^2)}{\frac{1}{2}\,|e^2+f^2|}\, d\theta$$

$$= \frac{1}{\ell(c)} \int_0^{2\pi} \delta e^2 + \delta f^2 \, d\theta,$$

while,

$$\ell(c) = \int_0^{2\pi} |c'(\theta)|\, d\theta = \frac{1}{2}\int_0^{2\pi} e^2(\theta) + f^2(\theta)\, d\theta$$

$$= \frac{2}{2} = 1, \quad (\text{since, } (e,f) \in S).$$

Thus

$$\langle h(\theta), h(\theta)\rangle_c = \int_0^{2\pi} \delta e^2 + \delta f^2 \, d\theta$$

$$= \langle (\delta e, \delta f), (\delta e, \delta f)\rangle_{L^2},$$

as required. $\square$

We have some obvious corollaries from the fact that Φ maps orthonormal 2-frames in ~~$L^2(I,\mathbb{R})^2$ to~~ $L^2([0,2\pi],\mathbb{R}^2)$ to closed curves:

Proposition. If $\Phi(e,f)=c$, and $(e,f)\in S^0$ then c is closed if and only if

$$(e,f)\in V_{even} \text{ or } V_{odd}.$$

$$\langle e,e\rangle=\langle f,f\rangle, \qquad \langle e,f\rangle=0.$$

Further, the winding number of c is even (odd) as (e,f) in V_{even}, V_{odd}.

Proof. The facts that c is (smoothly) closed $\Leftrightarrow (e(0),f(0))=\pm(e(2\pi),f(2\pi))$ and $\langle e,e\rangle=\langle f,f\rangle$, $\langle e,f\rangle=0$ are (by this point) expected consequences of the $z\mapsto z^2$ mapping.

Now the winding # of c is given by the number of times arg c' rotates around the unit circle.

Since this is <u>twice</u> the number of times the pair (e,f) circles the origin, this number is even iff (e,f) circles $(0,0)$ an integral number of times. $\square$.

Next, we compute

$$\kappa(\theta) = 2\,\frac{e f' - f e'}{(e^2+f^2)^2}.$$

To do this, we observe that

$$\kappa(\theta) = \frac{|c'(\theta) \times c''(\theta)|}{|c'(\theta)|^3}$$

But

$$c'(\theta) = (e(\theta) + i f(\theta))^2$$
$$c''(\theta) = 2(e(\theta)+if(\theta))(e'(\theta)+if'(\theta)).$$

Now for any vectors in $\mathbb{R}^2$ expressed as complex numbers, we see

$$(a+bi)\,i\,(c-di) = (a+bi)(d+ci)$$

$$= (ad-bc) + (ac+bd)i$$

$$= (a,b) \times (c,d) + (a,b)\cdot(c,d)\,i$$

So we can write

$$K(\theta) = 2\,\frac{\mathrm{Re}\left((e+if)^2\, i\, \overline{(e+if)}\,\overline{(e'+if')} \right)}{\left((e+if)^2\, \overline{(e+if)}^2 \right)^{3/2}}$$

$$= 2\,\mathrm{Re}\left(\frac{\cancel{(e+if)^2}\, i\, \cancel{\overline{(e+if)}}\,\overline{(e'+if')}}{(e+if)^{3}_{\uparrow 1}\,\overline{(e+if)}^{3}{}_{\nwarrow 2}} \right)$$

$$= 2\,\mathrm{Re}\left(\frac{i\,\overline{(e'+if')}}{(e+if)\,\overline{(e+if)}\,\overline{(e+if)}} \cdot \frac{(e+if)}{(e+if)} \right)$$

$$= 2\,\mathrm{Re}\left(i\,\frac{\overline{(e'+if')}\,(e+if)}{(e^2+f^2)^2} \right)$$

$$= 2\,\mathrm{Re}\left(i\,\frac{(e'-if')(e+if)}{(e^2+f^2)^2} \right)$$

$$= 2\,\mathrm{Re}\left(i\,\frac{(e'e+f'f)+(e'f-ef')i}{(e^2+f^2)^2}\right)$$

$$= 2\,\frac{ef'-fe'}{(e^2+f^2)^2}.$$

We then learn that

$$\kappa^2(\theta) = 2\,\frac{e^2(f')^2 - 2ef\,e'f' + f^2(e')^2}{(e^2+f^2)^4}.$$

Now if we're going to integrate

$$\int \kappa^2(\theta(s))\,ds = \int \kappa^2(\theta)\cdot|c'(\theta)|\,d\theta$$

$$= 2\int \frac{e^2(f')^2 - 2ef\,e'f' + f^2(e')^2}{(e^2+f^2)^2}\,d\theta$$

Hmm. $\frac{d}{d\theta}(e^2+f^2)^2 = 2(e^2+f^2)(2ee'+2ff')$

$\frac{d}{d\theta}(e^2+f^2) = 2ee'+2ff'$

$\cancel{e(\theta)=r(\theta)\cos\theta,\ f(\theta)=r(\theta)\sin\theta}$

$\cancel{ef'-fe' = r(\theta)r'(\theta)\cos\theta\sin\theta + r(\theta)^2\cos^2(\theta)} +$