The Plücker Embedding and Relations.

We now understand how to write $G_k(\mathbb{C}^n)$ as a subvariety of $\Lambda^k(\mathbb{C}^n)$ via the Plücker map

$$p(Y) = p(\text{span}(v_1, \ldots, v_k)) = v_1 \wedge \cdots \wedge v_k.$$

One way to characterize the image is as the completely decomposable vectors. We now want to go deeper into this ~~decomposition~~ subject to describe the decomposable vectors more explicitly. We first need to move to a more general Grassmannian.

Definition. If $\mathbb{C}^n$ consists of n-tuples $(a_1, \ldots, a_n)$, we can complete $\mathbb{C}^n$ to $\mathbb{P}^n$ by embedding

$\mathbb{C}^n$ in $\mathbb{C}^{n+1} - \vec{0}$ as the set $(1, a_1, \ldots, a_n)$, ~~and~~ adding the points $(0, b_1, \ldots, b_n)$ ~~and~~ (with b_i ~~#~~ not all 0) and identifying points related by scalar moltiplication.

As before, a linear subspace of $\mathbb{P}^n$ is the subspace obeying linear equations

$$\sum_{j=0}^{n} b_{\alpha j}\, p(j) = 0$$

for $\alpha \in 1, \ldots, (n-d)$. L is d-dimensional if the $(n-d) \times (n+1)$ matrix B has an ~~nonzero minor~~ $(n-d) \times (n-d)$ minor with nonzero determinant.

There are then $d+1$ points $P_i(0), \ldots, P_i(d)$ ~~of~~ in ~~$\mathbb{R}$~~ $\mathbb{P}^n$ which span L.

We want to represent this Grassmannian (either $(d+1)$-~~planes~~ subspaces in $\mathbb{C}^{n+1}$ ~~$\cong \mathbb{R}\ \mathbb{P}^n$~~ or ~~d-planes in $\mathbb{P}^n$~~ d-dimensional linear spaces ~~which~~ in ~~$\mathbb{R}$~~ $\mathbb{P}^n$ (which are NOT subspaces in $\mathbb{C}^n \subset \mathbb{P}^n$ but rather general linear subspaces not through the origin!) as a subspace of an $\binom{n+1}{d+1} - 1 = N$ dimensional projective space: $\mathbb{P}(\Lambda^{d+1}(\mathbb{C}^{n+1}))$

To do this, we establish coordinates on ~~$\mathbb{R}$~~ $\mathbb{P}^N$ called Plücker coordinates.

Idea: To any $(d+1)$ multindex j with $j_0, \ldots, j_d \in 0, \ldots, n$ and any $(n-d) \times (n+1)$ matrix A we can associate a number $p(j) = \det(\text{columns } j_0, \ldots, j_d \text{ of } A)$.

We call this

$$p(j_0, \ldots, j_d)$$

and note that this function is alternating on the j_i, and thus well-defined if we take the case $j_0 < j_1 < \ldots < j_d$ and extend to other orderings by ~~multilinearity~~. preserving the alternating property:

$$p(j_{\sigma(0)}, \ldots, j_{\sigma(d)}) = (-1)^{\operatorname{sgn} \sigma} p(j_0, \ldots, j_d).$$

We claim that the collection

$(p(j), \ldots, \cdot)$ for j a multindex of card. $d+1$ on $0, \ldots, n$

~~determines~~ for a matrix A representing a $d+1$ plane in $\mathbb{C}^{n+1}$ is invariant, under change of basis for the plane. (in $\mathbb{P}^N$)

If we change bases from A to B, $\exists$ some nonsingular $(d+1)\times(d+1)$ C ~~so which~~ so $CA = B$, or for each multindex j, the submatrices A_j, B_j obey

$$CA_j = B_j$$

so

$$(\det C)(\det A_j) = \det B_j$$

That is, the Plücker coordinates associated to the bases A and B are related by

$$\det C\,(p^A(j), \ldots) = (p^B(j), \ldots)$$

Now of course there are lots of points in $\mathbb{P}^N$ which aren't the values of an alternating function on multindices applied to a matrix.

Theorem. Suppose that j and K are multindices on $0,\dots,n$ where j has cardinality d and K has cardinality $d+2$. The Plücker coordinates of a linear subspace of a d-plane L in $\mathbb{P}^n$ obey

$$0=\sum_{\lambda=0}^{d+1}(-1)^{\lambda}\,p(j_0,\dots,j_{d-1},K_\lambda)\,p(K_0,\dots,\hat{K}_\lambda,\dots,K_{d+1})$$

Proof. As determinants, we are trying to show

$$\sum_{\lambda=0}^{d+1}(-1)^{\lambda}\begin{vmatrix} L_{j_0} & L_{j_1} & \cdots & L_{j_{d-1}} & L_{K_\lambda}\end{vmatrix}\begin{vmatrix} L_{K_0} & \cdots & \hat{L}_{K_\lambda} & \cdots & L_{K_{d+1}}\end{vmatrix}=0$$

We start by expanding the left determinants along their last column

$$\sum_{\lambda=0}^{d+1}(-1)^{\lambda}\left(\underbrace{\sum_{i=0}^{d}(-1)^{d+i}\begin{vmatrix}(\hat{L}_{j_0})_i & \cdots & (\hat{L}_{j_{d-1}})_i\end{vmatrix}}_{\text{doesn't depend on } \lambda}\underbrace{(L_{K_\lambda})_i}_{\text{a scalar}}\right)\begin{vmatrix}L_{K_0} & \cdots & \hat{L}_{K_\lambda} & \cdots & L_{K_{d+1}}\end{vmatrix}$$

so ~~this equals~~ we can reverse order of summation:

$$\left(\sum_{i=0}^{d}(-1)^{d+i}\begin{vmatrix}(\hat{L}_{j_0})_i & \cdots & (\hat{L}_{j_{d-1}})_i\end{vmatrix}\right)\sum_{\lambda=0}^{d+1}(-1)^{\lambda}(L_{K_\lambda})_i\begin{vmatrix}L_{K_0} & \cdots & \hat{L}_{K_\lambda} & \cdots & L_{K_{d+1}}\end{vmatrix}$$

Consider the right sum (for a fixed row i):

$$\sum_{\lambda=0}^{d+1}(-1)^{\lambda}(L_{K_\lambda})_i\begin{vmatrix}L_{K_0} & \cdots & (\hat{L}_{K_\lambda}) & \cdots & L_{K_{d+1}}\end{vmatrix} =$$ (expansion across first row of)

ith entry of all L_{K_λ}

$$= \begin{vmatrix} \leftarrow & (L_{K_\lambda})_i & \rightarrow \\ \uparrow & & \uparrow \\ L_{K_0} & \cdots & L_{K_{d+1}} \\ \downarrow & & \downarrow \end{vmatrix} = 0.$$

← = row i of matrix

These $(d+2)\times(d+2)$ matrices have a repeated ~~ith~~ row (top and position i) so all these determinants are zero, and the relation holds. $\square$

Theorem. Any point in ~~$\mathbb{R}$~~ $\mathbb{P}^N$ whose coordinates obey the Plücker relations comes from a unique d-plane L in $\mathbb{P}^d$.

Proof. Some ~~coord~~ coordinate of our point is nonzero, let it correspond to the multindex $K=(k_0,\dots,k_d)$.

We claim that ~~all of the remaining~~ since our point obeys the Plücker relations, all the $N+1$ coordinates of our point are uniquely determined by the coordinates of the form

$$p(k_0, \ldots, \hat{k}_\lambda, \ldots, k_d, j_\alpha)$$

~~where j_α~~ there are $(d+1)$ indices k_λ to take out and for each choice, $(n+1)-(d+1)$ remaining indices (not in K) to put in the j_α spot, ~~plus~~

If we put something in K back in, it must be that we deleted and put back k_d. So there are

$$(d+1)(n-d) + 1$$

coordinates of this form...