Schubert Calculus III

We have now proved:

Theorem. The Plücker relations determine the image of $G_k(\mathbb{C}^n)$ in $\mathbb{P}^N$ under the Plücker embedding.

We start with some examples.

Example. $G_2(\mathbb{R}^4)$.

There is only one sequence $j = (1)$ and $l = (234)$, which yields the Plücker relation

$$p(12)p(34) - p(13)p(24) + p(14)p(23) = 0.$$  

The sequences $j = (2)$, $l = (134)$ yield

$$p(21)p(34) - p(23)p(14) + p(24)p(13) = 0,$$

which is just

$$- p(12)p(34) - p(23)p(14) + p(24)p(13) = 0.$$
which is actually not distinct (it's just a change of sign from the previous).

If we view $C_a(\mathbb{R}^n)$ as the space of perimeter 2 polygons in $\mathbb{R}^2$ with 4 edges, we have the interpretation:

$$L = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$$

$$p(12) = \det [v_1, v_2] = v_1 \times v_2 = |v_1||v_2| \sin \Theta_{12}$$

$$p(ij) = \det [v_i, v_j] = v_i \times v_j = |v_i||v_j| \sin \Theta_{ij}$$

Now if we convert $v_i \rightarrow W_i$ via the Hopf map, we transform the angle $\Theta_{ij}$ to $\Phi_{ij} = 2\Theta_{ij}$, since the Hopf map is the complex squaring operation.
We can write the plücker relation as

\[ |v_1| |v_2| \sin \theta_{12} = |v_3| |v_4| \sin \theta_{34} - |v_1| |v_3| \sin \theta_{13} |v_2| |v_4| \sin \theta_{24} \]

\[ |v_1| |v_3| \sin \theta_{14} |v_2| |v_3| \sin \theta_{23} = 0 \]

We see that we can cancel \(|v_2||v_3||v_4||v_4|\) and get a relation

\[ \sin \frac{\phi_{12}}{2} \sin \frac{\phi_{34}}{2} - \sin \frac{\phi_{13}}{2} \sin \frac{\phi_{24}}{2} + \sin \frac{\phi_{14}}{2} \sin \frac{\phi_{23}}{2} = 0 \]

Some easy checks are:

- \(\phi_{13} = \pi\)
- \(\phi_{24} = \pi\)

So we get

\[ \sin \frac{\pi}{4} \sin \frac{\pi}{4} - \sin \frac{\pi}{2} \sin \frac{\pi}{2} + \sin \frac{3\pi}{4} \sin \frac{\pi}{4} = 0 \]

which is certainly true.
Harrison Chapman points out that we now rewrite these terms using

\[
\sin A \sin B = \frac{1}{2} \left( \cos (A - B) - \cos (A + B) \right)
\]

to get

\[
\frac{1}{2} \left[ \cos \left( \frac{\phi_{12} - \phi_{34}}{2} \right) - \cos \left( \frac{\phi_{12} + \phi_{34}}{2} \right) \right. \\
- \cos \left( \frac{\phi_{13} - \phi_{24}}{2} \right) + \cos \left( \frac{\phi_{13} + \phi_{24}}{2} \right) \\
\left. \cos \left( \frac{\phi_{14} - \phi_{23}}{2} \right) - \cos \left( \frac{\phi_{14} + \phi_{23}}{2} \right) \right] = 0.
\]

We now claim that since the \( \phi_{ij} \) are angles between vectors, we must have \( \phi_{ij} = -\phi_{ji} \) and

\[
\phi_{ij} + \phi_{jk} = \phi_{ik}
\]

with these relations, we have
\[ \phi_{12} - \phi_{34} = (\phi_{14} + \phi_{42}) - \phi_{34} = (\phi_{14} + \phi_{43}) + \phi_{42} = \phi_{13} - \phi_{24}, \]

so

\[ \cos\left(\frac{\phi_{12} - \phi_{34}}{2}\right) - \cos\left(\frac{\phi_{13} - \phi_{24}}{2}\right) = 0. \]

We then have

\[ \phi_{12} + \phi_{34} = (\phi_{13} + \phi_{32}) + \phi_{34} = (\phi_{13} + \phi_{34}) + \phi_{32} = \phi_{14} - \phi_{23}, \]

so

\[ -\cos\left(\frac{\phi_{12} + \phi_{34}}{2}\right) + \cos\left(\frac{\phi_{14} - \phi_{23}}{2}\right) = 0. \]

Lastly, we have

\[ \phi_{13} + \phi_{24} = (\phi_{12} + \phi_{23}) + \phi_{24} = (\phi_{12} + \phi_{24}) + \phi_{23} = \phi_{14} + \phi_{23}, \]

so

\[ \cos\left(\frac{\phi_{13} + \phi_{24}}{2}\right) - \cos\left(\frac{\phi_{14} + \phi_{23}}{2}\right) = 0. \]
The next question is how to extend this to space curves. We start by using the fact that the map from quaternions \( \eta : \text{Mat}_{2 \times 2}(\mathbb{C}) \) given by

\[
\eta(q) = \eta(a + bj) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}
\]

is a homomorphism. This means that if we take the matrix

\[
\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}
= M(q_1, q_2)
\]

then we have

\[
\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1
\]

Now if we take \( q_1 \overline{q}_2 \) as a quaternion product, we know that is given by

\[
\overline{q}_2
\]
\overline{q}_z = \overline{a}_z - b_2 \, j, \text{ so}

\begin{align*}
\overline{q}_z \overline{q}_2 &= \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} \begin{bmatrix} \overline{a}_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} \\
&= \begin{bmatrix} a_1 \overline{a}_2 + b_1 \overline{b}_2 & -a_1 b_2 + b_1 a_2 \\ -b_1 a_2 + a_1 \overline{b}_2 & a_2 \overline{a}_2 + b_1 b_2 + a_1 a_2 \end{bmatrix} \\
&= \langle (a_1, b_1), (\overline{a}_2, \overline{b}_2) \rangle - \mathbf{i} (a_1, b_1) \times (a_2, b_2) j \\
&= \langle v_1, v_2 \rangle - \text{det} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} j
\end{align*}

Now recall that as rotation matrices, quaternions are given by

\[ q = \cos \frac{\Theta}{2} + \mathbf{n} \sin \frac{\Theta}{2} \left( n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k} \right) \]

where \( \Theta \) is the angle of rotation and \( n_1, n_2, n_3 \) the axis of rotation. Thus as matrices

\[ \overline{q} = \cos \frac{\Theta}{2} - \sin \frac{\Theta}{2} \left( n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k} \right) \]
\[ = \cos(-\frac{\Theta}{2}) + \sin(-\frac{\Theta}{2})(n_1 i + n_2 j + n_3 k), \]

which is to say that \( \overline{q} \) is the inverse element of \( SO(3) \). Now

\[ q_2 \overline{a}_1 = \begin{bmatrix} a_2 & b_2 \\ -\overline{b}_2 & \overline{a}_2 \end{bmatrix} \begin{bmatrix} \overline{a}_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \]

\[ = \begin{bmatrix} a_2 \overline{a}_1 + b_2 b_1 & -a_2 b_1 + b_2 a_1 \\ -\overline{a}_1 b_2 + \overline{a}_2 b_1 & -\overline{b}_2 b_1 + \overline{a}_2 a_1 \end{bmatrix} \]

We can now observe that for non-unit quaternions, we have

\[ \text{det} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \frac{1}{2} q_1 ||q_2|| \sin \frac{\Theta_{12}}{2} (n_2 + n_3 i) \]

where \( \Theta_{12} \) is the angle of rotation.
But wait! There's a fly in the ointment! We have

\[
q_1 \quad q_2 \quad q_2 \overline{q_1} \rightarrow \cos \frac{\theta}{2} + \sin \frac{\theta}{2}(\eta_1 \text{i} \eta_2 \text{j} \eta_3 \text{k})
\]

But this doesn't mean that we can work backwards, since we can only determine \( \Theta \) from "downstairs" information to \( \Theta \pm 2\pi k \) and we would need to determine it in \([0, 4\pi)\) to reconstruct the quaternion \( q_2 \overline{q_1} \) unambiguously.

Equivalently

\[
\pm q_1 \xrightarrow{\text{Hopf}} F_1, \quad \pm q_2 \xrightarrow{\text{Hopf}} F_2
\]

so we can't know the sign of

\[
\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}
\]

by knowing \( F_1, F_2 \).
This means that we have really only proved

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \pm 1q_1 |l_q| \sin \frac{\Theta_1}{2} (n_2 + n_3 i),$$

and hence that there's a choice of signs which make the Plücker relation hold.

Which means the question remains open...