Schubert Calculus II

We ended last time with a Poincaré duality statement about Schubert cycles. Let us state it better.

Given a flag in $\underline{\mathbb{C}P^{n-1}}$ of linear subspaces

$$A_0 \subsetneq \dots \subsetneq A_{k-1} \quad \text{with dimensions } a_0, \dots, a_{k-1}$$

we can create a dual flag

$$A_{k-1}^{\perp} \subsetneq \dots \subsetneq A_0^{\perp} \quad \text{with dimensions } (n-1)-a_{k-1}, \dots, (n-1)-a_0.$$

Proposition. The Schubert cycles $\Omega(a_0, \dots, a_{k-1})$ and $\Omega((n-1)-a_{k-1}, \dots, (n-1)-a_0)$, are dual in the sense that

$$\Omega(a_0, \dots, a_{k-1}) \cup \Omega((n-1)-a_{k-1}, \dots, (n-1)-a_0) = \cancel{[G_k(\mathbb{C}^n;\mathbb{Z})]}:$$

(the generator of $H^{2k(n-k)}(G_k(\mathbb{C}^n);\mathbb{Z})$).

So what does this mean geometrically? Suppose we have two smooth, oriented submanifolds A and B of M, which intersect transversally in a submanifold $A \cap B$. We can choose an oriented basis for T_pM at any intersection point p

$$u_1, \ldots, u_{n-i-j}, v_1, \ldots, v_j, w_1, \ldots, w_j$$

so that $u_1, \ldots, u_{n-i-j}, v_1, \ldots, v_j$ is an oriented basis for T_pA, $u_1, \ldots, u_{n-i-j}, w_1, \ldots, w_j$ is an oriented basis for T_pB. Then we say

$$u_1, \ldots, u_{n-i-j} \text{ is an oriented basis for } T_p(A \cap B),$$

and this determines an orientation on all of $A \cap B$.

~~1~~ If we let

$$[X]^* \in H^{i}(M;\mathbb{Z})$$

be the Poincaré dual of the inclusion of the fundamental class of the $(n-i)$-dimensional oriented submanifold X, ~~then~~ denoted by $[X] \in H_{n-i}(M;\mathbb{Z})$, we have (see notes by Hutchings)

$$[A]^* \cup [B]^* = [A \cap B]^* \in H^{i+j}(M;\mathbb{Z}).$$

This is the basic theorem that we will use for the Schubert calculus.

We can conclude that if X is a smooth submanifold of $\cancel{\mathbb{CP}^{n-1}}$ $G_K(\mathbb{C}^n)$ then

$$[X]^* = \sum \delta((n-1)-a_{K-1},\dots,(n-1)-a_0)\,\Omega(a_0,\dots,a_{K-1}).$$

where $\sum a_i - i = K(n-K) - p$, and X has (real) codimension $2p$, or (real) dimension $2K(n-K)-2p$. Here the coefficients

$$\delta((n-1)-a_{K-1},\dots,(n-1)-a_0)$$

are given by

$$\big([X]^* \cup \Omega(n-1-a_{K-1},\dots,(n-1)-a_0)\big)\big([G_K(\mathbb{C}^n)]\big)$$

(by our theorem on cup products).

If X is transverse to the Schubert variety $\Omega((n-1)-a_{K-1},\dots,(n-1)-a_0)$ <u>and</u> that variety is a smooth manifold we may conclude that

$$\delta((n-1)-a_{k-1}, \dots, (n-1)-a_0)$$

is the number of points in

$$X \cap \Omega(B_0, \dots, B_{k-1})$$

where the B_i are linear spaces in a flag chosen so $\dim B_i = (n-1) - a_i$, counted with sign according to the orientations on X and $\Omega(B_0, \dots, B_{k-1})$.

Definition. The $\delta((n-1)-a_{k-1}, \dots, (n-1)-a_0)$ ~~are~~ with $\sum a_i - i = k(n-k) - p$ ~~is~~ are called the <u>degrees</u> of the subvariety X.

If Ψ is another irreducible subvariety of $G_k(\mathbb{C}^n)$ with re~~a~~al dimension $2p$, then it has degrees $\mathcal{E}$ so that

$$[\Psi]^* = \sum \mathcal{E}(a_0, \ldots, a_{k-1})\,\Omega((n-1)-a_{k-1}, \ldots (n-1)-a_0).$$

Then we have

Theorem. The number of intersection points in $X \cap \Psi$ is given by

$$\sum \mathcal{E}(a_0, \ldots, a_{k-1})\,\delta((n-1)-a_k, \ldots, (n-1)-a_0)$$

This is a sort of generalized Bezout's theorem: suppose we take the ambient space to be $G_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$. Then there is only ~~one~~ a_0 and a single degree for any subvariety.

So if

$$\dim X = 2(n-1) - 2p$$

then X has a single degree

$$[X^*]^* = \delta(p)\,\Omega(n-1-p)$$

and if

$$\dim Y = 2p$$

then Y has only a single degree

$$[Y]^* = \varepsilon(n-1-p)\,\Omega(p).$$

~~Further, since $\Omega(p)$ is a single~~

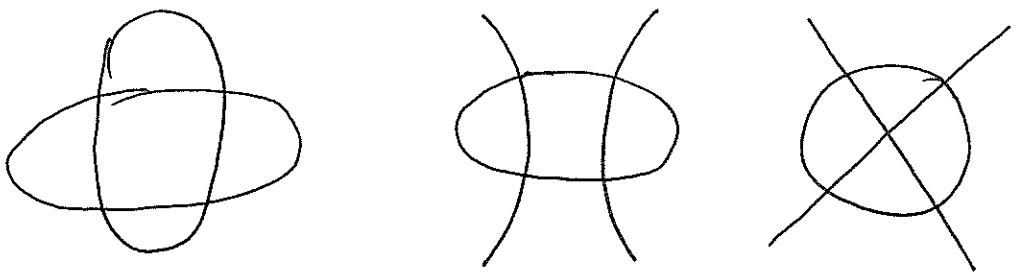
And

$$\# X \cap Y = \delta(p)\,\varepsilon(n-1-p).$$

We further observe that $\Omega(p)$ is a single p-dimensional linear subspace of $\mathbb{CP}^{n-1}$, so the degree of X is the degree of the homogenous polynomial defining it when X is a hypersurface.

Example. $G_1(\mathbb{C}^3) \cong \mathbb{CP}^2$.

If we let X and Y be defined by quadratic polynomials, we see that



the number of intersections is 4 (when transverse).

We now have that the cohomology group of a Grassmannian in any dimension may be expressed as a linear combination of Schubert cycles.

Can we get an even smaller generating set using the cup product?

Definition. The special Schubert cycle $\sigma(h) = \Omega(h, n-K+1, \dots, n-1)$, where $h \in 0, \dots, n-K$.

Example. In $G_3(\mathbb{C}^7)$ we have the special Schubert cycles $\sigma(0), \dots, \sigma(4)$ given by

$$\sigma(i) = \Omega(i, 5, 6).$$

We then have

The Determinantal Formula.

For all sequences of integers $0 \leq a_0 \leq \dots < a_{K-1} < n$ we have

$$\Omega(a_0, \dots, a_{d-1}) = \begin{vmatrix} \sigma(a_0) & \sigma(a_0 - 1) & \cdots & \sigma(a_0 - (K-1)) \\ & & \vdots & \\ \sigma(a_{K-1}) & \sigma(a_{K-1} - 1) & & \sigma(a_{K-1} - (K-1)) \end{vmatrix}$$

where by convention $\sigma(h) = 0$ for $h < 0$ or $h > n-K$.

Example. For $G_2(\mathbb{C}^n)$, we have the special Schubert cycles

$$\sigma(0) = \Omega(0, n-1), \ldots, \sigma(n-2) = \Omega(n-2, n-1)$$

and the general Schubert cycle

$$\sigma(a_0 a_1) = \begin{vmatrix} \Omega(a_0, n-1) & \Omega(a_0-1, n-1) \\ \Omega(a_1, n-1) & \Omega(a_1-1, n-1) \end{vmatrix}$$

$$= \Omega(a_0, n-1) \cup \Omega(a_1-1, n-1) - \Omega(a_1, n-1)\,\Omega(a_0-1, n-1).$$

Now this means that the cup product of any pair of Schubert cycles can be reduced to the case where one is a special Schubert cycle.

In this case, we have

Theorem (Pieri's Formula).
For all sequences $0 \le a_0 < \ldots < a_{K-1} < n$ of integers and $h \in 0, .., n-K$, we have

$$\Omega(a_0, .., a_{K-1}) \cup \sigma(h) = \sum \Omega(b_0, .., b_{K-1})$$

where the sum covers all sequences b_i

$$0 \le (b_0 \le a_0) < (b_1 \le a_1) < (b_2 \le a_2) < \ldots < (b_{K-1} \le a_{K-1}) < n$$

with $\sum b_i = \sum a_i - (n-K-h)$.

We now revisit the question of counting lines intersecting 4 lines in $\mathbb{C}P^3$.
We saw before that if we let

$$Q = \bigcap \Omega(L_i, \mathbb{C}P^3),$$

Then we needed to count the points in Q.

This must be given by $\Omega(1,3)^4$.

Now $\Omega(1,3) = \sigma(1)$ among the special Schubert cycles of $G_2(\mathbb{C}^4)$ so

$$\Omega(1,3) \cup \sigma(1) = \sum \Omega(b_0, b_1)$$

where $0 \leq b_0 \leq 1 < b_1 \leq 3$, and $b_0 + b_1 = 4-(2-1) = 3$.

There are two such sequences,

$(0,3)$ and $(1,2)$,

so

$$\Omega(1,3)^2 = \Omega(0,3) + \Omega(1,2).$$

~~Now $\Omega(0,3)$ is already~~ a special Schubert ~~cycle, so~~

$$\cancel{\Omega(0,3) \cup \sigma(0) = \sum \Omega(b_0, b_1)} \qquad \cancel{3-(2-0)=1.}$$

~~where $0 \leq b_0 \leq 0 < b_1 \leq 3$ and $b_0 + b_1 =$~~ $\cancel{1}$,

~~This is gives $\Omega(0,3)^2 = \Omega(0,1)$.~~

So

$$\Omega(1,3)^3 = \Omega(0,3) \cup \sigma(1) + \Omega(1,2) \cup \sigma(1).$$

Using Pieri's formula again,

$$\Omega(0,3) \cup \sigma(1) = \sum \Omega(b_0, b_1)$$

where $0 \le b_0 \le 0 < b_1 \le 3$, $b_0 + b_1 = 3-(4-2-1) = 2$,

or

$$\Omega(0,3) \cup \sigma(1) = \Omega(0,2)$$

and

$$\Omega(1,2) \cup \sigma(1) = \sum \Omega(b_0, b_1)$$

where $0 \le b_0 \le 1 < b_1 \le 2$, $b_0 + b_1 = 3-(4-2-1) = 2$.

We again get

$$\Omega(1,2) \cup \sigma(1) = \Omega(0,2)$$

so

$$\Omega(1,3)^4 = 2\Omega(0,2) \cup \sigma(1)$$

Using Pieri one last time we see

$$\Omega(0,2) \cup \sigma(1) = \sum \Omega(b_0, b_1)$$

So that $0 \leq b_0 \leq 0 < b_1 \leq 2$, $b_0 + b_1 = 2 - (4-2-1) = 1$
which means

$$\Omega(0,2) \cup \sigma(1) = \Omega(0,1),$$

and

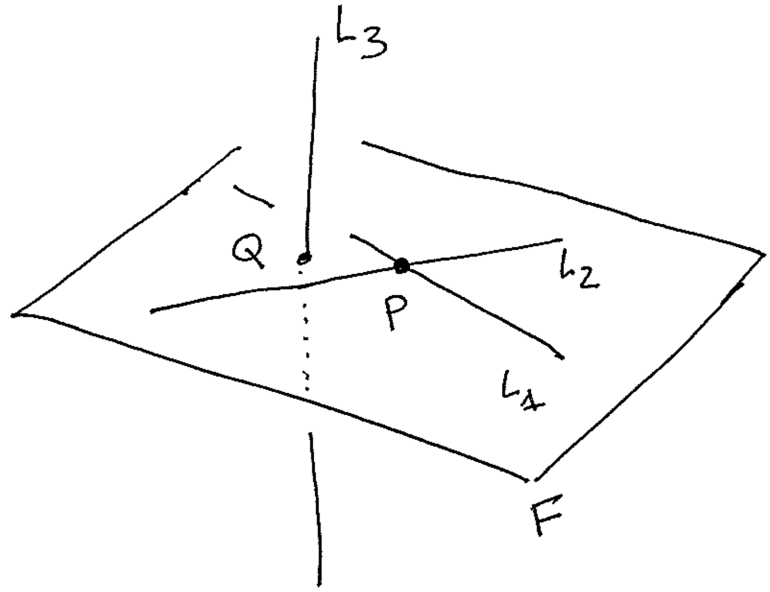
$$\Omega(1,3)^4 = 2\,\Omega(0,1).$$

Since $\Omega(0,1)$ generates the top dimensional cohomology group $H^8(G_2(\mathbb{C}^4);\mathbb{Z})$ we see that the intersection $Q = \bigcap \Omega(L_i, \mathbb{C}P^3)$ generally has two points!

Indeed, we saw more: since

$$\Omega(1,3)^3 = 2\,\Omega(0,2) = 2\left(\text{generator of } H^6(G_2(\mathbb{C}^4;\mathbb{Z})\right),$$

We expect the family of lines through 3 given lines to ~~have~~ consist of 2 families of lines with complex dimension 1. We can see this by specialization:



Any line through Q in the plane F <u>or</u> any line through P and L_3 intersects all 3 lines.