Holonomy and Gauss-Bonnet (II).

We have now learned that
\[ \int_{\alpha} \Phi_{12}(s) \, ds = - \iint_{R} K \, d\text{Area} \]

Now, we know that if
\[ \alpha'(s) = \cos \Theta e_1 + \sin \Theta e_2 \]

(remember - everything on the rhs depends on \( s \)) that

\[ K_g(s) = \Phi_{12} + \Theta' \]

so for a closed curve \( \alpha(s) \) bounding a simply connected region \( R \) in the \( uv \) plane

\[ \int_{\alpha} K_g(s) \, ds = \int_{\alpha} \Theta'(s) \, ds - \iint_{R} K \, d\text{Area}. \]

Now we can see that if the
curve has length $L$, then $\alpha'(0) = \alpha'(L)$

$$\int_0^L \theta'(s)ds = 2\pi k, \text{ for } k \in \mathbb{Z}.$$  \(\Box\)

But what is $k$? Well, if we let $\alpha$ shrink to a point continuously then we can see that we're measuring the winding number (in the $uv$-plane), which is $+1$ (once around the circle), so $k = 1$.

Thus we have:

**Theorem.** If $R$ is a simply connected region in the $uv$ plane, with $\partial R$ smooth

$$\oint_{\partial R} K \, ds + \iint_R K \, dA = 2\pi.$$
Proof. Approximate the curve by smooth curves and note that the total geodesic curvature of a rounded off corner converges to the exterior angle. □

Corollary. For a geodesic triangle with interior angles $\phi_1, \phi_2, \phi_3$,

$$\int K \, d\text{Area} = \sum \phi_i - \pi.$$  

Proof. The edges are geodesics, and $\phi_i = \pi - \Theta_i$, so

$$\int K \, d\text{Area} + \sum \Theta_i = 2\pi$$

$$\Rightarrow \int K \, d\text{Area} = 2\pi - \sum \pi - \phi_i = \sum \phi_i - \pi. \quad \square$$
Example. Any curve with total geodesic curvature zero on the sphere (bounds area $2\pi$) divides the sphere into equal pieces.

Example of example. A great circle!

We can extend our theorem to curves with corners by defining turning angle as the (oriented) angle between tangents at a corner.

Theorem. If $R$ is parametrized by a simply connected region with $\partial R$ having corners, then with turning angles $\Theta_1, \ldots, \Theta_n$,

$$\int_R \kappa_g \, ds + \int \int_R K \, d\text{Area} + \sum \Theta_i = 2\pi.$$
Spherical cap. Now it works!

Define triangulation.

$X$ definition.

Gauss-Bonnet.

$$\int_M K_g \, ds + \iint_M KdArea + \sum_{\Sigma \subseteq M} \epsilon_k = 2\pi X$$

Write down triangle sum, approximate sums of tetrahedra exterior angles to get $E, V$ terms.

Conclusions.