More on symmetry and majorization.

Suppose

$$(x + a_1) \ldots (x + a_n) = x^n + C_1 x^{n-1} + \ldots + C_n$$

$$= x^n \binom{n}{1} P_1 x^{n-1} + \binom{n}{2} P_2 x^{n-2} + \ldots + P_n$$

The $C_i$ are the elementary symmetric functions of the $a_i$—that is

$$C_1 = a_1 + \ldots + a_n$$

$$C_2 = a_1 a_2 + a_1 a_3 + a_1 a_4 + \ldots = \sum_{i < j} a_i a_j$$

$$C_k = \sum_{i_1 < i_2 < \ldots < i_k} a_{i_1} a_{i_2} \ldots a_{i_k}$$

$$C_n = a_1 \ldots a_n$$

while the $p_i$ are the corresponding averages (instead of sums).
All of the $p_i$ are symmetric means of the $a_i$, since

$$C_r = \left( \sum \frac{1}{r! (n-r)!} \right) \frac{1}{\text{ways to rearrange } n-r \text{ terms left out of product}}$$

each of these occurs multiple times

and

$$P_r = \frac{1}{\binom{n}{r}} C_r = \frac{1}{\frac{n!}{r! (n-r)!}} C_r = \frac{1}{n!} \sum a_1 \ldots a_r = \underbrace{[1,1,\ldots,1,0,\ldots,0]}_{r \text{ times}}$$

Now if $p_r = [a]$, then $\sum a_i = r$, so different $p_r$ can’t majorize one another and so can’t be directly comparable. They can (and do!) obey non-linear inequalities.

Theorem (Newton) $P_{r-1} P_{r+1} \leq P_r^2$, for any real $a_i$ (including negative ones!), with equality only if all $a_i$ are equal.
\[
\frac{n!}{(n-(i+j))!} \frac{(n-(i+j))!}{(n-(i+j)-(k-j))!(k-j)!} P_k x^{n-k-i} y^{k-j}
\]

Now let's pick some \( r \) and let \( i = n-r-1 \), \( j = r+1 \), so \( i+j = n-2 \). Then most of the \( x^{n-k-i} \) and \( y^{k-j} \) vanish. Simplifying, the general term is

\[
\frac{n!}{2!} \left( \frac{2}{k-r+1} \right) P_k x^{r+1-k} y^{k-r+1}
\]

and this is nonzero only when

\[
k-r+1 = 2, \quad k-r+1 = 1, \quad k-r+1 = 0
\]
Lemma. If
\[ f(x, y) = c_0 x^m + c_1 x^{m-1} y + \ldots + c_m y^m = 0 \]
has all roots \( x/y \) real, then same is true of all roots of
\[ \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x, y) = 0. \]
(unless \( \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x, y) \) vanishes, identically.)

Further, if \( \alpha \) is a root of multiplicity \( M > 1 \) of one of these equations, it is a root of multiplicity \( M+1 \) of the equation \( f' \) for which \( \frac{\partial}{\partial x} E' = E \) or \( \frac{\partial}{\partial y} E' = E \).

Proof. Exercise.
Suppose that
\[ f(x,y) = (x + a_1 y) \cdots (x + a_n y) \]
\[ = p_0 x^n + \binom{n}{1} p_1 x^{n-1} y + \cdots + \binom{n}{n} p_n x^0 y^n \]
\[ = \sum_{k=0}^{n} \binom{n}{k} p_k x^{n-k} y^k \]

We compute
\[ \frac{\partial^i+j}{\partial x^i \partial y^j} \binom{n}{k} p_k x^{n-k} y^k = \]
\[ = \binom{n}{k} p_k (n-k)(n-k-1)\cdots(n-k-i+1) K(k-1)\cdots(k-j+1)x^{n-k-i} y^{k-j} \]
\[ = \binom{n}{k} p_k \frac{(n-k)!}{(n-k-i)! (k-j)!} x^{n-k-i} y^{k-j} \]
\[ = \frac{n!}{k! (a^k)!} p_k x^{n-k-i} y^{k-j} \]